"High Watermarks of Market Risks"

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Abstract

The volatility has long been used as an auxiliary variable in the processes explaining the returns on risky assets. In this traditional framework, the observable were the returns and the volatility remained a latent variable, whose value or possible values were a by-product of the estimation. Recently, the focus has changed and many studies have been devoted to empirical estimates of the volatility itself, without specifying necessarily any model for the prices themselves.

This has been made possible by the increased availability of high-frequency data, and the theoretical works of Barndorff-Nielsen and Shephard (2002) showing convergence between an empirical measure of volatility and its theoretical expression. The empirical measure of volatility has been progressively refined, from a simple sum of squared returns to more sophisticated measures taking into account microstructure biases (see for instance Domen, 2005). In parallel, some theoretical developments have put back into focus the role of jumps. Taking the volatility as a random variable in itself means studying its characteristics. It is well known that volatility dynamics are autoregressive but also that obviously its process is stationary. Given that, it is natural to look for the best fit for the distribution of the volatility, given that the theory yields several possible candidates. Of special interest is the estimation of the likelihood of the volatility peaks, which relies on Extreme Value Theory.

In this article, we first present several estimates of measures of risk, using both high frequency data and lower frequency data. The aim here is also to show what lower frequency measures can be substitute to the high precision measures when transaction data is unavailable. The second part is devoted to the studies of the distribution of the volatility, using general forms of common distribution functions. Finally, we focus on the slope of the tail of the various risk measure distribution, in order to estimate the frequency of extreme events and define the high watermarks for market risks. Using several techniques of estimation, we finally do not find evidence for the need of a specification with heavier tails than the ones of the traditional log-normal. The tail estimates additionally yield return times for the extreme market events, as another reality check.

Keywords: Financial Crisis, Realized Volatility, Bi-power Variation, Range-based Volatility, Extreme Value, High Frequency Data.

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1. Introduction

Measure and allocation of risk is central to the theory and practice of finance. Market risk has been identified with various measures over time, but the most widely used by practitioners remains today the volatility, a time-varying simile of the traditional variance.

Following for instance Barndorff-Nielsen and Shephard (2003) or Andersen et al (2003), volatility can be viewed as a latent factor (the so-called quadratic variation affecting the Brownian motion in some representations) that can only be estimated using its signature on market prices. It is only when the process is known (and simulated) as in Andersen and Bollerslev (1997, 1998) or Barndorff-Nielsen and Shephard (2002-a) that we know what the “true volatility” is (inside the Brownian motion case - see Andersen et al, 2003, for a survey). When the underlying process is more sophisticated as shown by Barndorff-Nielsen and Shephard (2002-b), or when observed prices suffer from market microstructure distortions effects (see Corsi et al, 2001, Andersen et al, 2000, Bai et al, 2001, Oomen, 2002, Brandt and Diebold, 2003), the results are less clear.

Realized volatility has been, since its introduction (see Andersen and Bollerslev, 1998-a) considered the best estimator for the latent factor. The daily volatility retrieved from transaction data has been shown to be accurate when controlling for microstructure effect and empirically supporting the Clark (1973) Mixture of Distribution Hypothesis (Andersen et al, 2000). Among the high-frequency estimators, the one using all the available transactions (VARHAC estimator, Bollen and Inder, 2002) performs better than the realized volatilities that use a lower sampling rate. Oomen (2005) empirically shows that estimating the volatility in business-time (transaction-time) is more efficient than using the traditional calendar-time, as it samples the process when it is the most informative. Aït-Sahalia et al. (2005) show that the most accurate estimator, so far, is the mean of the realized volatilities chosen at the optimal frequency but measured at different phases.

When high-frequency data is unavailable, second best estimations of the unobservable risk factor are provided by the range-based – or extreme value – estimators. The price range, defined as the difference between the highest and lowest market prices over a fixed sampling interval, is known for a long time as a volatility estimator. From Parkinson (1980), there is a wealth of literature' devoted to refinements of this measure – using various assumptions on the underlying process.

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The aim of the present article is to study high and low-frequency measures of volatility, in order to find if there are glaring discrepancies between the empirical evidence and the usual assumptions on the distribution of volatility. In the first part of this article, we will focus on various estimates of volatility on the French CAC40 index, using both transaction data and range-based estimators. The second part is devoted to the study of the distributional properties of some of the various volatility estimators, testing the goodness-of-fit of several candidate probability distributions. Finally, in the third part, we fit Extreme Value distributions to the right-hand tail of daily risk measures in order to get estimated frequencies of high watermarks and return times of extreme events.

2. Realized and Range-based Daily Measures of Risk

It has been known since the first attempts at modelling prices that the amplitude of price changes was not constant, but fluctuating with time in a somewhat predictable fashion. Though it is not possible to estimate the spot volatility (i.e., the value of the volatility at a point in time), the integrated variance (i.e., the variance of the instantaneous returns over a period) can be approximated through estimators of the quadratic variation (the limit of the sum of squared returns over decreasing time intervals). We compute hereafter measures of daily volatility, both from intra-day data (which should converge to quadratic variation in the absence of jumps as per Barndorff-Nielsen and Shephard, 2003) and daily data (the range-based estimators should be less sensitive to outliers and jumps than the squared returns). Following Andersen et al. (2005), we also compute the jump part of the realized volatility, by removing the bi-power variation from the realized volatility.

The first sub-section will present the main measures of risk, both high-frequency and extreme value. The second sub-section will likewise introduce similar asymmetric measures of risk. Finally, we present an illustration of the various measures over a sample of CAC40 30" quotes on the period March 1995 - February 2005.

2.1 Symmetric Measures of Risk and Extreme Value Estimators of Volatility

In many financial applications, return dispersions are estimated by calculating the standard deviation of a time series of daily, weekly, or monthly returns, based on a security’s closing prices. If the risk was deemed a constant over the time, then the accuracy of the variance would increase with the number of points used in its computation. Since market risk fluctuates with time, using past observations in the current measure can actually be detrimental if those belonged to a different distribution. One way to cope with the problem is to use a rolling window and compute the classical variance over a fixed period of time. Typical time periods vary from one year to about five business days, depending on the investment horizon and actual use of the data (risk control, volatility trading or marketing purposes). The traditional proxy of the volatility, is thus, with daily data:
\[
\hat{\sigma}_t = \left[ \frac{1}{(N-1)} \sum_{n=t-N}^{t} \left[ \ln\left( \frac{P_n}{P_{n-1}} \right) - \hat{\mu}_t \right]^2 \right]^{1/2}
\]

where \( N \) is estimation window expressed in number of business days, dates \( t \) correspond to ends of business days, \( \{P_n\} \) is a sequence of closing prices and:

\[
\hat{\mu}_t = \frac{1}{N} \sum_{n=t-N}^{t} \left[ \ln\left( \frac{P_n}{P_{n-1}} \right) \right]
\]

is an estimation of the mean log-return on the reference period. This daily estimate will thus exhibit by construction lots of serial dependence, since the same returns observations enter several consecutive volatilities. As pointed by Poon and Granger (2002), since the statistical properties of sample mean make it a very inaccurate estimate of the true mean especially for small sample, taking deviations around zero - or around a very long period mean - instead of the sample mean increases the accuracy of the estimate even if biased. Taking this reasoning to the extreme, the simplest measure of volatility would be given by the squared return between two sampling dates (days), that is:

\[
\hat{\sigma}_t^{\text{simple}} = \left( \eta \ln\left( \frac{P_t}{P_{t-\tau}} \right) \right)^{1/2}
\]

where \( \eta = n_b / \tau \) is the annualizing factor, \( n_b \) being the number of business days per year and \( \tau \) the periodicity (one day by default), and \( \{P_t\} \) is the series of the price of the asset at time \( t \). This measure is free of hypothesis on the mean return, and does not use overlapping information: we will thus use it as our instantaneous low frequency volatility in the rest of the paper. It is however extremely noisy, and not to be used in practical applications.

This classical standard deviation of return series is assumed to be a proxy for the future dispersion of returns. Though it is easy to compute, it ignores all the information concerning the path of the price inside the period of reference. Even at the low (daily) frequency, though, additional information is often available in addition to the closing price, such as opening price and highest and lowest prices within the day. Feller (1957) has derived the joint distribution of the maximum and minimum of a random walk, assuming no drift and constant dispersion. This result has been used by Parkinson (1980), who proposes an estimator of the variance for the process based on the range of the random walk (available as the difference between highest and lowest price) during the day considered. Parkinson (1980) shows that the additional information provided by the high/low records of the period improves the accuracy of the estimation. His estimator is a function of the range and reads (with the previous notations):

\[
\hat{\sigma}_t^P = \left[ \frac{1}{\theta N} \sum_{n=1}^{N} \left[ \ln\left( \frac{H_n}{L_n} \right) \right]^2 \right]^{1/2}
\]
where:
\[
H_n = \text{Arg}_{t_i} \max \{ P_{t_i} \mid t_i \in [n-1, n] \}
\]
is the highest price at day \( n \)
\[
L_n = \text{Arg}_{t_i} \min \{ P_{t_i} \mid t_i \in [n-1, n] \}
\]
is the lowest price at day \( n \)
\[
\theta_N = 4N \ln(2)
\]
is a correction parameter.

The Parkinson’s (1980) extreme value estimator efficiency intuitively comes from the fact that the range of intra-daily quotes gives more information regarding the future volatility than two arbitrarily spaced points in this series (the closing prices), for the low cost of two additional data points per day. Most data suppliers already provide daily high/low as summaries of intra-day activity. Roughly speaking, knowing these records allows us to get closer to the “real underlying process”, even if we do not know the whole path of asset prices.

But using all four data points - open, close, high and low prices - instead of two - close-to-close or high-low prices - can also give extra information - especially if opening and closing prices are followed by all market participants and act as references. Garman and Klass (1980) propose an estimator based on the knowledge of the open, close, high and low prices that can be written (with previous notations):

\[
\hat{\sigma}_{\text{GK}}^2 = \left[ \alpha_1 \sum_{t-N}^{t} \ln(H_n / L_n)^2 - \alpha_2 \sum_{n-t-N}^{t-N} \ln(C_n / C_{n-1})^2 \right]^{1/2}
\]

where \( C_n \) is the closing price at day \( n \), \( \alpha_1 \) and \( \alpha_2 \) are weighting parameters\(^2\).

Since Parkinson’s (1980) and Garman-Klass’ (1980) estimators implicitly assume that log-stock prices follow a geometric Brownian motion with no drift, further refinements are given by Rogers and Satchell (1991) and Kunitomo (1992). The latter author uses the open and close prices to estimate a modified range corresponding to a hypothesis of a Brownian bridge of the transformed log-price. This basically aims to correct the high and low prices for the presence of a drift (with previous notations):

\[
\hat{\sigma}_K^2 = \left[ \frac{1}{\beta_N} \sum_{p=t-N}^{t} \ln(h_n / l_n)^2 \right]^{1/2}
\]

where:

\(^2\) with \( \alpha_1 = \frac{5}{N} \) and \( \alpha_2 = \frac{39}{N} \).
\[
\begin{aligned}
\hat{H}_n &= \text{Arg}_{i_n} \left\{ \text{Max}_{P_{i_n}} \left[ O_n - \left( C_n - O_n \right) / t_i + \left( C_n - O_n \right) \right] \right\} \\
&= \text{Arg}_{i_n} \left\{ \text{Max}_{P_{i_n}} \left[ O_n + \left( C_n - O_n \right) / t_i - \left( C_n - O_n \right) \right] \right\} \\
&= \text{the end - of - the - day projection of drift - correceted highest price at day } n \\
\hat{L}_n &= \text{Arg}_{i_n} \left\{ \text{Min}_{P_{i_n}} \left[ O_n + \left( C_n - O_n \right) / t_i + \left( C_n - O_n \right) \right] \right\} \\
&= \text{the end - of - the - day drift - correceted lowest price at day } n \\
\beta_n &= 6 \left( N \pi^2 \right) \text{ is a correction parameter}
\end{aligned}
\]

Rogers and Satchell (1991) also add a drift term in the stochastic process that can be incorporated into a volatility estimator using only daily open, high, low, and closing prices, that reads (with previous notation):

\[
\hat{\sigma}_{RS}^2 = \frac{1}{N} \sum_{n=1-N}^{t} \left[ \ln(H_n/O_n) \ln(H_n/O_n) - \ln(C_n/O_n) \right] + \ln(L_n/O_n) \ln(L_n/O_n) - \ln(C_n/O_n) \right] \right)^{1/2}
\]

where \( O_n \) is the open price at day \( n \).

They also propose an adjustment that is designed for taking into account the fact that one may not be able to continuously monitor the stock price. Their adjusted estimator is the positive root of the following quadratic equation:

\[
\frac{1}{I_n} \sum_{n=1-N}^{t} \left( \frac{0.5594}{\hat{\sigma}_{RS}^2} + \frac{0.9072}{\sqrt{I_n}} \ln(H_n/L_n) \hat{\sigma}_{ARS}^2 + \hat{\sigma}_{RS}^2 \right) = 0
\]

with \( I_n \) is the total number of transactions occurring during day \( n \).

Finally, Yang and Zhang (2000) make further refinements by deriving an extreme-value estimator that is unbiased, independent of any drift, and consistent in the presence of opening price jumps. Their estimator thus writes (with previous notation):

\[
\hat{\sigma}_{YZ}^2 = \left[ \frac{1}{N-1} \sum_{n=1-N}^{t} \left( \ln(O_n/C_{n-1}) - \ln(O_n/C_{n-1}) \right)^2 + \frac{1}{N-1} \sum_{n=1-N}^{t} \left( \ln(C_n/O_n) - \ln(C_n/O_n) \right)^2 \right]^{1/2}
\]

where:

\[
\kappa = \frac{0.34}{1.34 + \frac{N+1}{N-1}}
\]

with \( \sigma \) being the unconditional mean of \( x \) and \( \hat{\sigma}_{RS}^2 \) being the Rogers-Satchell (1991) estimator (see above text).

The Yang-Zhang’s estimator is simply the square root of the sum of the estimated overnight variance (the first term on the right hand side) and the estimated open market variance (which is a weighted average of the open market
return sample variance) and the Rogers and Satchell (1991) drift independent estimator (where the weights are chosen so as to minimize the variance of the estimator). The resulting estimator therefore explicitly incorporates a term for the overnight variance.

While the extreme estimators were still dealing with traditional measures, the availability of tick-by-tick data led to a reframing of both theoretical and empirical literature on volatility. Instead of considering a constant volatility over a certain period of time (a day for instance), the continuous time model assumes a continuously varying volatility. The risk to be approximated over the period considered is thus no longer a constant value, but the so-called integrated volatility.

More formally, in the continuous framework, we get:

\[ d \log(P_t) = \mu \, dt + \sigma_t \, dB_t, \]

with \( P_t \) the price at time \( t \), \( \mu \) the drift term, \( B_t \) the standard Brownian motion and \( \sigma_t \) the instantaneous volatility. The quantity of interest is here the integrated volatility over the time interval \( \tau \), that is \( \int_0^\tau \sigma_t^2 \, dt \).

In this framework, the empirical integrated volatility is in fact the realized quadratic variation defined as:

\[ \sigma_{t,\tau}^2 = \sum_{j=2}^{t/\tau} \left( \ln \left( \frac{P_j}{P_{j-1}} \right) \right)^2, \]

where \( \tau \) is the time interval between observations and \( \{P_j\} \) the sequence of high frequency (intraday) prices. Though this estimator converges in theory towards the integrated volatility as the periodicity of observation goes down, in practice microstructure effects skew the price dynamics, meaning that the either the empirical estimation has to be done on a lower frequency than the maximum available or it has to incorporate correcting terms, or both (see, among many others, Oomen, 2002 and 2005 for a discussion of this point).

Pushing further the analysis, Barndorff-Nielsen and Shephard (2004) have recently introduced the so-called realized bi-power variation, which writes, with the previous notations:

\[ B_{PV_{t,\tau}} = \sum_{j=2}^{t/\tau} \ln \left( \frac{P_j}{P_{j-1}} \right) \cdot \ln \left( \frac{P_{j-1}}{P_{j-2}} \right), \]

The realized bi-power variation can be used to split the realized variance between its continuous and discontinuous (jump) components. The intuition is that while squaring the returns amplifies the jumps, the bi-power variation, by multiplying each jump with the preceding or following return, dampens their overall effect. The difference between realized volatility and bi-power variation will hereafter be called jump component and used to quantify the intensity of the jumps within one day.
The variance measures can be encompassed with the following formula for the empirical standard deviation computed at time $t$ (end of the window of the $\rho$ time-periods $\tau$) and at a time-scale $\tau$ (end of the window of the period $\rho$), which reads $^3$:

$$\sigma_{\tau} = \left[ \sum_{i=1}^{I} \ln \left( \frac{P_{i} / P_{i+1}}{\eta_{\sigma}} \right) \right]^{1/2}$$

where $\eta = n_{b} / \tau$ is a factor $^4$ for annualizing the time-scale volatility, $n_{b}$ is the number of business days per year, $\tau$ is the time-resolution expressed in number of observations per day, $\{P_{i}\}, i = [1, ..., I]$, is the sequence of prices of the asset at time $t$, with $t_i = t - i \tau / I = [1, ..., I]$, the date of observation of return involved in the time-scale $\tau$ volatility and $\Delta_{\tau} = \tau / I$ is the time-increment depending on the time scale.

As crucially emphasized Alizadeh et al (2002), range-based estimators have many attractive properties over either low frequency estimators or even, for some authors, high-frequency based volatility estimators (Aït-Sahalia et al, 2005, Marten and Van Dijk 2006).

The range is a highly efficient volatility proxy as shown by Brandt and Diebold (2003) in a multivariate setting. In turbulent days with drops and recoveries of the markets, the traditional close-to-close volatility can still be low while the daily range correctly indicates that the volatility is high. Furthermore, the range appears robust to microstructure biases such as the bid-ask bounce. As pointed by Brandt and Diebold (2003), the observed daily maximum is general reached at the ask price and hence is too high by half the spread, whilst the observed minimum is likely to be attained at the bid price and therefore is too low by half the spread. Generally speaking, the bid-ask bounce inflates thus the range only by the average spread, which is small in general in liquid markets relative to the price moves. The price range is however inefficient when compared to high-frequency estimators (see Andersen and Bollerslev, 1998, p. 898, footnote 20, and Andersen et al, 2001) in the Brownian motion case. But it is far from clear that range-based estimators are less efficient outside the Brownian case study or in the presence of market microstructure effects, simply because the problem is difficult to handle theoretically and because microstructure noise covers a large collection of potential biases which are difficult either to properly define or explain (see Crack and Ledoit, 1996).

In terms of efficiency (measured as the ratio of the variance of the extreme value estimator over the close-to-close one), all previous estimators exhibit very substantial improvements. As highlighted in Corrado and Miller (2006), Parkinson (1980) reports a theoretical relative efficiency gain ranging from 2.5 to 5, while the Garman and Klass

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$^3$ Note that we do not apply the correction mentioned by Figlewski (1998) to remove the bias in the standard deviation estimation since it is proven to be very small in sample (see Poon and Granger, 2003).

$^4$ Note that, by contrast to Martens (2002), Areal and Taylor (2002) and Hol and Koopman (2002), we do not correct here the volatility for the presence of noisy overnight returns since we do not focus on a special time-scale accurate measure, but rather on a specific time-series observation scale corresponding to economic agent observation frequency.
Yang and Zhang (2000) and Kunitomo (1992) variance estimators result in a theoretical efficiency gain of, respectively, 7.4, 7.3, and 10. Rogers et al. (1994) report that the Rogers-Satchell’s estimator yields theoretical efficiency gains comparable to the Garman-Klass’ estimator. They also report that the Rogers-Satchell estimator performs well with changing drift and as few as 30 daily observations.

Consistent with the previous research of Wiggins (1991), Li and Weibaum (2002) find that the Parkinson’s estimator - applied to daily S&P 100 and S&P 500 data covering periods of 1 to 24 (trading) days over the period from January 1989 to December 1999 - is downward biased compared to the traditional estimator, at the weekly and monthly frequencies. This is also true for the other extreme value volatility estimators. They provide rationales to explain this result and other coherent extreme value estimates on S&P 500 Futures and exchange rates markets (see Li and Weibaum, 2000, p.19 and following). Furthermore, Corrado and Miller (2006) conclude that when used in Merton’s (1980) Reward-to-Risk model of expected excess return, most of extreme estimators of volatility lead to similar qualitative results. Corrado and Truong (2006) also demonstrate that the intraday High-Low estimator price range offers volatility forecast with efficiency similar to Implied volatility as predictor of future volatility. Extreme estimators finally seem to be highly correlated one to another and strongly linked to the implied and future volatilities.

The set of Figures I represents the various weekly estimates - namely Realized Volatility, Classical Empirical Variance, Parkinson (1980), Garman and Klass (1980), Rogers and Satchell (1991), Kunitomo (1992), Yang and Zhang (2000) Extreme Value Variance estimations - of daily volatility, using 30’ CAC40 French stock index intraday quotes, resampled at a 30’ frequency on the period 03-1991/02-2005. These numbers are square root of variance estimates, annualized by multiplying it by the square root of the number of trading days per annum divided by the number of days in its volatility interval (see Hull, 2000). The peaks of the variance estimates are approximately synchronous, but the general behaviour of the series differs, both in the range of variances and persistence phenomenon (see section 3).
The close-to-close volatility appears extremely irregular and noisy, whereas the realized volatility is smoother, and displays the phenomenon of persistence of variance far more clearly. The Parkinson’s estimator is closer in behaviour to the realized volatility, whereas the additional terms in Kunitomo’s formula actually seem to make the estimator less regular and in particular produces more very-low volatilities, not unlike the basic close-to-close squared return.

The following graph represents the main measures of volatility plotted against the realized volatility (used as benchmark). The scatter plot reveals that the main difference between the measures lies in their accuracy, since no bias appears in the slopes of the graph, but the range of possible values is much larger for the close-to-close and Kunitomo volatilities.

Figures 2: Daily Estimates of Simple and Extreme Volatilities

versus Realized Volatility


2.2 Alternative Asymmetric Measures of Risk

When returns are skewed and leptokurtic, better measures of volatility have been discussed in the literature. For instance, semi-variance estimator only focuses on the left part of the return distribution that corresponds to true losses for investors. The rescaled\(^5\) mean semi-standard deviation reads (with previous notations):

\[
\hat{\sigma}^{SV}_t = \left[ \frac{2}{(N-1)} \sum_{n=t-N}^{t} [\text{Min}(P_n/P_{n-1}) - \bar{\mu}_t, 0] \right]^{2\frac{1}{2}}
\]

while its DownSide Risk mean empirical counterpart equals to:

\[
\hat{\sigma}^{DSR}_t = \left[ \frac{2}{(N-1)} \sum_{n=t-N}^{t} [\text{Min}(P_n/P_{n-1}), 0] \right]^{2\frac{1}{2}}
\]

This adaptation is particularly efficient when extreme variations are asymmetric. Using an asymmetric measure of risk better characterizes the drop in the market and does not signal periods of booms where semi-volatility

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\(^5\) We multiply the semi-variance (and DownSide Risk measure) by two since, if the underlying density is symmetric, the semi-variance is equal to half the variance. Taking the square root of this rescaled estimator give then a number that is of the same order that standard deviation and mean of returns.
is low.\(^6\) Noting furthermore that the high/low ratio used in the Parkinson’s estimator (see text above) is close to a classical risk measure called Drawdown, we substitute the Drawdown to the high/low ratio into Parkinson’s formula to get a mean squared Drawdown measure\(^7\) of risk reading (with previous notation):

\[
\hat{\sigma}_{\text{MDD}_t} = \left[ \frac{1}{\gamma_N} \sum_{n=t-N}^{t} \left[ \ln \left( \frac{H_n}{L_n^n} \right) \right]^2 \right]^{1/2}
\]

where:

\[
L_n^* = \arg \left\{ P_t \mid t_j > t, (t_i, t_j) \in [n-1, n] \right\}
\]

is the lowest price following the highest price at day \(n\)

\(\gamma_N\) is a scaling parameter

By using the Drawdown measure instead of Parkinson’s one, we focus on negative returns in order to assess crises. The Drawdown quantifies the financial losses in a conservative way: it compute the losses of the worst investment - from highest price to lowest - in the reference period. This approach reflects quite well the preferences of investors who define their allowed losses in percentages of their initial investments. While an investor may excuse a short-term Drawdown on his account, he would definitely start worrying about his capital in the case of a long-lasting Drawdown. Such a Drawdown may indicate that something is wrong with a specific market and maybe it is time to retrieve the money and place it in a more successful investment pool. So a Drawdown accounts not only for the amount of losses, but also for the duration of these losses. As pointed by Krokhal et al (2002), the Drawdown is a loss measure “with memory” taking into account the magnitude of losses but also the time sequence of losses: a long range of small negative returns will entail a high drawdown, characterizing a decreasing trend in prices with a low volatility\(^8\).

For stressing the extreme variation of prices, we can also substitute - being even more conservative - the maximum to the average drawdown in the previous formula that now reads:

\[
\hat{\sigma}_{\text{MDD}_t} = \left[ \frac{1}{\lambda_N} \left[ \ln \left( \frac{H^{**}_t}{L^{**}_t} \right) \right]^2 \right]^{1/2}
\]

with:

\(^6\) Note, however, that the highest period volatility generally corresponds to a first large decrease in prices followed by a second recovering in prices. In other words, the largest moves in the markets are drops in price, and not booms (see Goodhart and Danielsson, 2002 and Maillet and Michel, 2003).

\(^7\) Note that this definition differs from the \(\alpha\)-CDaR (mentioned in Krokhal et al, 2002) in the sense that first it is the unconditional mean of recent drawdowns (on the window estimation) and second it does not depend upon a (conditional) threshold; note also that we slightly modify the traditional definition of drawdown (see Johanssen and Sornette, 2001) since they correspond here to the worst investment timings within the estimation window. That allows avoiding the difficult problem of \textit{ex ante} determining the highest and lowest price on the reference period.

\(^8\) As experienced at the end of May and beginning of June 02 where stock markets - measured by a World Index such as the MSCI World for instance - was down regularly at a small rate on a daily basis. The volatility - measured on a short window - was low during this period since dispersion of returns around the mean was also low. Using the Drawdown measure instead of the classical allows for making the risk measure more adequate to what is intuitively a crisis.
\[
H_{t}^{**} = \text{Arg}\left[ \text{Max}_{i \in [t-N,t]} \{ P_i \} \right]
\]
is the highest price on the period \([t - N, t]\)

\[
L_{t}^{**} = \text{Arg}\left[ \text{Min}_{i \in [t, t+N]} \{ P_i \} \right]
\]
is the lowest price following the highest price on the period \([t - N, t]\)

\(\lambda_N\) is rescaling parameter

An illustration of these measures on the French market is provided in Figure 3.

**Figure 3: Daily Estimates of Annualized Asymmetric Risk Measures**

a. Realized Semi-volatility  
b. Drawdown

![Figure 3: Daily Estimates of Annualized Asymmetric Risk Measures](image)

Source: Euronext; 30’ resampled intraday CAC40 French stock index quotes on the period 03-01-1995/02-28-2005. Computations by the authors. Realized Semi-volatility (left) and Drawdown (right) are presented in this figure.

The Realized Semi-volatility (drawdown) is analogue to the Realized (respectively Parkinson’s) Volatility. Both estimates appear somewhat more noisy than their symmetric counterparts, but less so than the simplest volatility measures.

### 2.3 Descriptive Statistics and Correlations

Table 1 presents the higher moments of the empirical log-volatilities. The asymmetry coefficient is mostly negative (except for Parkinson’s); the mass of probability in the right side of the distribution does not appear significantly larger than on the left side. The kurtosis differs across measures, with Parkinson’s and the Realized Volatility appearing close to the Gaussian, while the Close-to-Close’s and Kunitomo’s measures are more heavy-tailed.

**Table 1: Statistics of the Log-Volatilities**

<table>
<thead>
<tr>
<th>Volatility Estimators</th>
<th>Empirical Skewness</th>
<th>Empirical Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Realized</td>
<td>-0.03</td>
<td>3.43</td>
</tr>
<tr>
<td>Close-to-close</td>
<td>-1.14</td>
<td>5.28</td>
</tr>
<tr>
<td>Parkinson</td>
<td>0.21</td>
<td>3.04</td>
</tr>
<tr>
<td>Kunitomo</td>
<td>-1.22</td>
<td>6.81</td>
</tr>
<tr>
<td>Drawdown</td>
<td>-0.09</td>
<td>2.87</td>
</tr>
</tbody>
</table>

Source: Euronext; 30’ resampled intraday CAC40 French stock index quotes on the period.
Overall, as already seen in Figure 1, estimators using intra-day data are less volatile (more accurate) than the classical estimator.

The following table corresponds to the correlation matrices of risk log-estimations.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Realized</td>
<td>1.00</td>
<td>0.45</td>
<td>0.86</td>
<td>0.67</td>
<td>0.76</td>
<td>0.25</td>
<td>0.75</td>
<td>0.80</td>
<td>0.93</td>
</tr>
<tr>
<td>Close-to-close</td>
<td>0.35</td>
<td>1.00</td>
<td>0.55</td>
<td>-0.09</td>
<td>0.28</td>
<td>0.21</td>
<td>0.37</td>
<td>0.33</td>
<td>0.28</td>
</tr>
<tr>
<td>Parkinson</td>
<td>0.93</td>
<td>0.43</td>
<td>1.00</td>
<td>0.75</td>
<td>0.79</td>
<td>0.29</td>
<td>0.78</td>
<td>0.65</td>
<td>0.73</td>
</tr>
<tr>
<td>Garman-Klass</td>
<td>0.53</td>
<td>-0.26</td>
<td>0.61</td>
<td>1.00</td>
<td>0.73</td>
<td>0.18</td>
<td>0.64</td>
<td>0.52</td>
<td>0.49</td>
</tr>
<tr>
<td>Kunitomo</td>
<td>0.59</td>
<td>0.17</td>
<td>0.73</td>
<td>0.60</td>
<td>1.00</td>
<td>0.22</td>
<td>0.74</td>
<td>0.62</td>
<td>0.65</td>
</tr>
<tr>
<td>Rogers-Satchell</td>
<td>0.24</td>
<td>0.12</td>
<td>0.26</td>
<td>0.16</td>
<td>0.22</td>
<td>1.00</td>
<td>0.58</td>
<td>0.51</td>
<td>0.30</td>
</tr>
<tr>
<td>Drawdown</td>
<td>0.58</td>
<td>0.20</td>
<td>0.70</td>
<td>0.53</td>
<td>0.67</td>
<td>0.59</td>
<td>1.00</td>
<td>0.87</td>
<td>0.70</td>
</tr>
<tr>
<td>Realized Semi-vol.</td>
<td>0.73</td>
<td>0.18</td>
<td>0.56</td>
<td>0.41</td>
<td>0.53</td>
<td>0.51</td>
<td>0.84</td>
<td>1.00</td>
<td>0.82</td>
</tr>
<tr>
<td>Jump Component</td>
<td>0.93</td>
<td>0.28</td>
<td>0.73</td>
<td>0.49</td>
<td>0.65</td>
<td>0.3</td>
<td>0.70</td>
<td>0.82</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Source: Euronext, 30" sampled intraday CAC40 French stock index quotes on the period 03-01-1995/02-28-2005. Computations by the authors. Empirical Pearson (upper triangle) and Spearman (lower triangle) correlation coefficients between risk measures.

Parkinson’s volatility is very close to the realized volatility and thus can be used as a proxy when all the data is not available intra-daily. Another intraday measure (namely Kunitomo’s one) is also highly correlated with these two, in contrast with Garman-Klass’ and Rogers-Satchell estimators. Lastly the Close-to-Close squared return evolution is relatively poorly correlated with all other measures. These results are confirmed by the analysis of the relative rankings of the observed market sessions according to their volatility (Spearman correlation between measures).

3. The Distribution of Volatilities

It is now generally admitted since the seminal papers by Cizeau et al (1997) and Andersen et al (2001-a and 2001-b) that the log-volatility is approximately Gaussian for a daily integrated-horizon. This hypothesis is nevertheless challenged both for extremely high and low frequencies. The unconditional ultra high-frequency return density is leptokurtic, due to the presence of fat tails but also to a peak around zero-return (no-event) in calendar-time; the density of the volatility at these frequencies should be then degenerated with a high level of probability around zero (zero-return, zero-volatility). Unconditional low frequency return density is by contrast approximately almost Gaussian, and the density of the volatility at a low frequency should then be close to a Chi-square. As underlined by Forsberg and Bollerslev (2002), this time-aggregation effect should be a characteristic feature of volatility modelling. An unrelated problem is the amplitude of some extreme events, specifically the 1929 and 1987 crises which defy most attempts to
assign a probability to such huge variations. Miccichè et al. (2002) show that the right tail of a log-normal density underestimates the probability of the most extreme realizations of the volatility. For Forsberg and Bollerslev (2002), the Normal Inverse Gaussian is a reasonable distribution for the variance - accounting for main induced scale-relation parameters - in the Mixing Distribution Hypothesis of Clark (1973). Bontemps and Meddahi (2005) recently confirm quite intensively this finding.

3.1 Distributional Properties of Volatilities

Many articles have been dedicated to distribution of financial returns (see for instance McDonald, 1994), but relatively few on the distribution of empirical volatility (see, for instance, Andersen et al., 2001 or Thomakov and Wang, 2003). Distributional properties of risk measures are important features for our purpose of measuring the financial crisis. Several papers advocate that log-volatility is normally distributed (see Andersen et al., 2001) while others show that Beta, Pearson type V or stretched exponential densities (see below) exhibit better fits depending on the measure, the data and the time-scale. Indeed, as it has been shown for raw returns, one might expect the shape of the volatility distribution to be deformed when increasing the time scale resolution. For instance, it is well known that ultra high-frequency data exhibit a very sharp peak around zero due to the parsimony of information arrival at this time scale.

Natural candidates for representing risk estimation distribution are the following:

• the log-normal (see, for instance, Cizeau et al., 1997 or Andersen et al., 2001):

\[
F_{ln} (\sigma) = \frac{1}{\hat{\sigma}_{\sigma} \sqrt{2\pi}} \int_{s=0}^{\sigma} \exp \left\{ \frac{-[\ln(s) - \hat{\mu}_{\sigma}]^2}{2\hat{\sigma}_{\sigma}^2} \right\} ds
\]

with \( \hat{\mu}_{\sigma} \) and \( \hat{\sigma}_{\sigma} \) being the empirical mean and standard deviation of the volatility.

• the more general and flexible Beta distribution (see Johnson and Kotz, 1995 and Kotz, 2002):

\[
F_{B} (\sigma) = \int_{s=0}^{\sigma} s^{\alpha} (1-s)^{\beta} B(\alpha, \beta) ds
\]

with the scaling Beta function such as:

\[
B(\alpha, \beta) = \int_{\sigma=0}^{1} \sigma^{\alpha} (1-\sigma)^{\beta} d\sigma \quad \text{for} \quad (\alpha, \beta) \in IR^+^2.
\]

• the Pearson type V (or Inverted Gamma, see Miccichè et al., 2002):

\[
F_{PV} (\sigma) = 1 - F_{B}(1/\sigma)
\]

where \( F_{B}(.) \) is the Beta distribution.

The Hull-White hypothesis (see Hull and White, 1987):

\[
F_{HW} (\sigma) = 2 \left( \frac{ba/\xi^2}{\Gamma(1+a/\xi^2)} \right)^{1+1/\xi^2} \int_{s=0}^{\sigma} \left[ \frac{\exp\left( -ba/\xi^2 \right)}{s^{2a/\xi^2+3}} \right] ds
\]
• the transformed Hull-White hypothesis (see Miccichè et al., 2002):

\[
F_{MBLM}(\sigma) = \frac{(ba/\xi^2)^{1/2}}{\Gamma(1 + a/\xi^2)} \int_0^\infty \frac{\exp(-ba/\xi^2 s)}{s^{a/2+1}} \, ds
\]

where the corresponding probability density function is the Inverted Gamma:

\[
f_{IG}(\sigma) = \frac{B^C}{\Gamma(C)} \frac{\exp(-B/\sigma)}{\sigma^{C+1}}
\]

where \( B = \frac{ba}{\xi^2} \) and \( C = 1 + \frac{a}{\xi^2} \).

• the Stretched Exponential (Johanssen and Sornette, 2001):

\[
F_{SE}(\sigma) = \frac{1}{\beta} \left[ \exp\left(-\frac{\sigma}{\beta}\right)\right]^{-\alpha}
\]

with the scale parameter being \( \beta = 1^{1/\alpha} b^{1/\alpha} \), \( b \) a constant and \( \alpha > 1 \).

• The Inverse Gaussian (Barndorff-Nielsen, 1997):

\[
f_{IG}(\sigma) = \frac{\alpha}{\sqrt{2\pi}\beta} \sigma^{\frac{3}{2}} \exp\left(-\frac{(\alpha - \beta \cdot \sigma)^2}{2\beta\sigma}\right)
\]

for \( \sigma > 0 \), and 0 otherwise.

Some other forms of distributions – more general would also be advocated such as the Generalized Hyperbolic Distribution that encompasses the NIG and the Skew Student’s t (see for instance Barndorff-Nielsen and Stelzer, 2004). Its probability density reads:

\[
f_{HGD}(\sigma) = \frac{\gamma^{\nu - 1/2 - \nu}}{\sqrt{2\pi} \, \sqrt{\delta K_{\nu}(\gamma)}} \left(1 + \frac{(\sigma - \mu)^2}{\delta^2}\right)^{-1/2 - 1/4} K_{\nu - 1/2}\left(\frac{\sigma - \mu}{\sqrt{1 + (\sigma - \mu)^2/\delta^2}}\right) \exp[\beta(x - \mu)].
\]

with \( \gamma = \sqrt{\alpha^2 - \beta^2}, \, \tilde{\alpha} = \delta\alpha, \, \tilde{\beta} = \delta\beta, \, \tilde{\gamma} = \delta\gamma \) and where \( K_{\nu}(.) \) denotes the modified Bessel function of the third kind of order \( \nu \).

The Generalized Gamma would also be a fair candidate since it encompasses the Gamma, the Inverted Gamma and the Weibull distributions. Its density function reads:

\[
f_{GG}(\sigma) = \frac{1}{\Gamma(k)} \frac{\beta}{\alpha} \left(\frac{\sigma}{\alpha}\right)^{\beta - 1} \exp\left[-\left(\frac{\sigma}{\alpha}\right)^\beta\right],
\]

with \( \alpha \) the scale parameter, \( k \) the index parameter and \( \beta \) the power parameter.
All scale and shape parameters are estimated hereafter using matching moment maximum likelihood estimation method (see, for instance, Law and Kelton, 1991, for the underlying conditions) and cumulative density functions performed using numerical approximations. Estimating general forms of these density functions allows us to discriminate between the various candidates based on the parameters of the general form.

As an example, the next figure illustrates different hypotheses for the distribution of the Parkinson volatility.

**Figure 4: An Illustration of Empirical Estimation of Volatility Density Function**

The Log-normal density seems to underestimate the right tail of the volatility (under interest here) as does the Normal Inverse Gaussian. The most visible differences lie in the volatility mode (around 10%), where the NIG seems to give a better fit.

Table 3 presents the results of goodness-of-fit tests for all measures, for all the sampling frequencies and all distributions considered (preliminary version – section under progress, to be completed).
Table 3: P-values of a Goodness-of-fit Test against the Log-normality Hypothesis

<table>
<thead>
<tr>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P-Values</td>
<td>0.37</td>
<td>0.00</td>
<td>0.62</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Source: Euronext; 30’ resampled intraday CAC40 French stock index quotes on the period 03-01-1995/02-28-2005. Computations by the authors. Are reported in this table the empirical P-values of a Kolmogorov-Smirnov Goodness-of-fit test.

The goodness-of-fit test confirms the diagnosis from the higher moments: Gaussianity cannot be rejected at traditional significance levels for the more accurate estimates (namely Realized and Parkinson’s Volatility and Drawdown), while the presence of the heavy tails mean that normality is clearly rejected for all other measures at a 5% confidence threshold.

4. Extreme Values of the Daily Risk Estimates

The goodness-of-fit tests used in the previous section yields use the information of the whole range of observations. A rejection can thus be caused by differences in the left hand tail of the data or the mode and, likewise, small local differences in the curve can be diluted in the whole sample. However, a diagnosis of market volatility is relevant in turbulent periods and some inaccuracy in the low-risk periods can be tolerated provided the estimator performs best otherwise. When studying volatility distribution, the area of interest is thus the right-hand tail where the highest volatilities are located. The following sections of the article are accordingly devoted to the exploration of the rightmost part of the distribution function in order to highlight the differences between the estimators.

According to the central result of Extreme Value Theory, the extrema of the measures of the risk should converge asymptotically to the Generalized Extreme Value (GEV) distribution. This distribution is characterized by the parameter known as the tail index, denoted $\xi$ (which is the opposite of the inverse of the shape parameter of the distribution), measuring the rate of decreases of the probability in the tails, such as the Jenkinson limiting distribution of an IID normalized minimum $\sigma$, for $\sigma \in IR$, written:

$$H_\xi(\sigma) = \begin{cases} \exp\left[-(1 + \frac{\xi}{\alpha})(\frac{h + \sigma}{\alpha})^{-\frac{1}{\xi}}\right] & \text{if } \xi \neq 0 \\ \exp\left(-\frac{(h + \sigma)^{-\xi}}{\alpha}\right) & \text{otherwise} \end{cases}$$

where $h$ and $\alpha$ are location and scale parameters and $\xi$ is the so-called tail index, with $(1 + \xi \sigma) > 0$ and $\xi \in IR$.

Three basins of attraction can be distinguished according to the value of the tail index $\xi$: positive, zero or negative. In the case of fat-tailed distributions (Gnedenko Limiting distribution of Fréchet for the Extremes and a leptokurtic distribution for the underlying Log-Volatilities), the tail index will be significantly positive, whereas the
Gaussian underlying distribution yields a null tail index (corresponding to the Gumbel distribution for the Extremes and the Normal for Log-volatilities).9

Extreme Value Theory in its original definition deals with independent observations, whereas the persistence of variance is a well known phenomenon in finance, meaning the extremes will occur in clusters. Nevertheless, providing that the dependence is weak, the previous results hold in the sense that resulting distribution is also a GEV, intensified by a power real - called the extremal index. Thus GEV distributions for IID and non-IID series theoretically have the same tail index, because rising to a power the GEV distribution only affects its scale and location parameters (see Byström, 2004). Some empirical experiments yet conclude to the possible impact on the non-respect of the IID hypothesis (Kearns and Pagan, 1997). When the GPD does not perfectly fits the data and in a context of parameter uncertainty, this entails taking some precautions when interpreting results. Still, Coles (2001) suggest that the Block-Maxima method (BM Method) - considering sub-samples longer that the potential serial dependency for estimating GEV - should reduce the impact on the parameter uncertainty.

Since the GEV distribution needs to be estimated on the maxima of the underlying random variable, which comprise necessarily a relatively small sample, the risk of misspecification is quite important. Several methods, parametric and non-parametric can be used to retrieve the value of the tail index; for the sake of robustness we present hereafter different estimates for the various estimators under studies.

Figure 5 shows the non-parametric tail estimator by Pickands (1975), plotted against the number of points included in the tail (that is, the \( k \) highest daily volatilities within the sample). The estimator should in theory converge to its true value as the number of points increases, but is biased when points from the centre of the distribution are included. The tail index appears negative for most of the measures (namely clearly for the Realized and the Parkinson’s ones), except for Kunitomo’s and Rogers-Satchell’s ones.

---

9 Intuitively, the third case where the tail index is null - which corresponds to the Weibull-kind of distributions for Extremes and platikurtic ones for Log-volatilities should be \textit{a priori} off the scope of our analysis dedicated to extreme risks.
The non-parametric Hill (1975) estimator is more accurate, but is designed for the estimation of positive extreme value indexes. It writes:

$$H_{k,n} = \frac{1}{k} \sum_{i=1}^{k} \log S_{n-i+1} - \log S_{n-k}$$

where $n$ is the number of observations, $k$ is the number of points in the tail and $S_{nj}$ is the order statistics of order $j$. Beirlant and al. (1996) proposed an extension of this estimator to any value of the extreme value index. This extension write, using the previous notations:

$$GH_{k,n} = \frac{1}{k} \sum_{i=1}^{k} \log S_{n-i} - \log S_{n-k-1} + \frac{1}{k} \sum_{i=1}^{k} \log H_{i,n} - \log H_{k+1,n}$$

The first term is the classical Hill estimator with the deletion of the maximum of the series and the second term is computed on the classical Hill estimators themselves.
Here again, the extreme value indices appear to converge, all stabilizing to a negative value as soon as the estimation includes at least the 25 upper volatilities.

Amongst the existing parametric methods, the most common practice for backing-out the relevant parameters of the GEV is to use a direct numerical maximization of the Log-likelihood. Alternatively, one can also use the estimation of empirical counterparts of L-moments for assessing the tail index as underlined hereafter.

The L-moments, which are linear functions of the expectations of order statistics, were introduced by Sillitto (1951) and comprehensively reviewed by Hosking (1990). L-moments have found wide applications in such fields of applied research as meteorology, hydrology and regional analysis (see Hosking, and Wallis, 1997).

They are defined as linear functions of the probability-weighted moments; they have been shown to provide more robust estimators of higher moments that the traditional sample moments (Sankarasubramaniani and Srinivasan, 1999), and, in our case, easy to compute and reliable estimators for extreme distributions.

Formally, the L-moment of order \( r \) is defined as:

\[
\lambda_r = \sum_{k=1}^{r} p_k^* \beta_k
\]

Despite their superiority on the sample moments, some authors nevertheless still advocate the instability of L-moment and propose robust alternatives - see the UQ-moments by Mudholkar and Hutson, (1998), or the TL-moments by Elamir and Seheult, (2003).
where $p^*$ are the Legendre polynomials coefficients, and $\beta_k$ can be estimated without bias from the sample probability weighted moment computed such as:

$$\hat{\beta}_k = \frac{1}{n} \left( \binom{n-1}{k} \right)^{-1} \sum_{i=1}^{n} \binom{i-1}{k} x_{n(i)}$$

where $x_{n(i)}$ is the $i$-th order statistics.

In practice, the sample $r$-th standardized higher order L-moments are often used, defined as $\hat{\tau}_r = \hat{\lambda}_r / \hat{\lambda}_2$ with $r > 2$.

The parameters of the GEV can be directly retrieved from the sample empirical moments (Ben-Zvi and Azmod, 1997) or from sample L-moments (Hosking et al, 1985). When using the L-moments, matching the parameters of the distribution to the population L-moments gives the following equation for the tail parameter:

$$\hat{\xi} = -7.86 \cdot \hat{z} - 2.96 \hat{z}^2$$

with $\hat{z}$ is a function of the standardized sample third L-moment, denoted $\hat{\tau}_3$, such as:

$$\hat{z} = 2 \hat{\tau}_3^{-1} - \ln(2)/\ln(3).$$

Mudholkar and Hutson (1998) refine the L-moment by replacing the expectation by the quick estimate of the location parameter of the order statistics. Thus, their sample LQ moments is defined, as before, from the Legendre polynomials as:

$$\varsigma_r = \sum_{k=0}^{r-1} p^*_{k,r} \tau(X_{r-k,r})$$

where $p^*$ are the Legendre polynomials coefficients, and $\tau(X_{k,r})$ is a quick estimate of the location parameter of the order statistics $X$. Mudholkar and Hutson (1998) propose the median, trimean and Gastwirth estimator for practical application, we use here the median of the order statistics, as estimated with the inverse beta distribution. With the median, for instance, the quick estimator is equal to: $\tau(X_{r-k,r}) = Q_{x}(B_{r-k,r}^{-1}(\frac{1}{2}))$, with $Q_x$ the quantile function and $B_{r-k,r}^{-1}()$ the inverse Beta function. The resulting LQ moments of order 1 to 3 are needed to estimate the GEV parameters, starting with the shape parameter.
Table 4: Summary of GEV Parameters Identification by Moment Matching

<table>
<thead>
<tr>
<th>Order</th>
<th>Moments</th>
<th>L-Moments</th>
<th>L-Q Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \mu = h - \frac{\alpha}{\xi}(1 - \Gamma(1 - \xi)) )</td>
<td>( \lambda_1 = h - \frac{\alpha}{\xi}(1 - \Gamma(1 - \xi)) )</td>
<td>( \xi_1 = h + \frac{\alpha}{2} \left( \frac{Q_0(\frac{1}{4}) + Q_3(\frac{3}{4})}{2} + \frac{Q_3(\frac{1}{2})}{3} \right) )</td>
</tr>
<tr>
<td></td>
<td>( \sigma^2 = \left( \frac{\alpha}{\xi} \right)^2 \left( (1 - 2\xi) - \Gamma^2(1 - \xi) \right) )</td>
<td>( \lambda_2 = \alpha \left( 1 - 2^{-\xi} \right) \frac{\Gamma(1 - \xi)}{\xi} )</td>
<td>( \xi_2 = \frac{\alpha}{8} \left( 2Q_0(0.7) - 2Q_0(0.29) + Q_3(0.87) \right) )</td>
</tr>
<tr>
<td>3</td>
<td>( s = \frac{\Gamma(1 - 3\xi) - 3\Gamma(1 - 2\xi)\Gamma(1 - \xi) + 2\Gamma^3(1 - \xi)}{\left( \Gamma(1 - 2\xi) - \Gamma^2(1 - \xi) \right)^{1/2}} )</td>
<td>( \xi_3 = -7.86 \cdot \hat{z} + 2.96 \hat{z}^2 )</td>
<td>( \xi_2 = 2P(Q_0(u)) ) *</td>
</tr>
<tr>
<td></td>
<td>( \hat{z} = 2\hat{z}_3^{-1} - \ln(2)/\ln(3) )</td>
<td>( Q_0(u) = -\frac{1 - (-\log(u))^{-\xi}}{\xi} )</td>
<td></td>
</tr>
</tbody>
</table>

* the coefficients of the polynomial \( P \) are given in Mudholkar and Hutson (1998)

As presented in the following Table 5, none of the estimated tail indexes, estimated on maxima of the daily volatilities, is positive, meaning once again that none of the underlying distributions can be considered fat-tailed.

Table 5: Estimate of Tail Indexes of Generalized Extreme Value Distributions of Period Maxima of Daily Log-volatilities via Moments Matching Method

<table>
<thead>
<tr>
<th></th>
<th>Realized</th>
<th>Jump Component</th>
<th>Close-to-close</th>
<th>Parkinson</th>
<th>Garman-Klass</th>
<th>Kunitomo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weekly</td>
<td>-0.43</td>
<td>-0.42</td>
<td>-0.23</td>
<td>-0.44</td>
<td>-0.31</td>
<td>-0.39</td>
</tr>
<tr>
<td>Monthly</td>
<td>-0.34</td>
<td>-0.47</td>
<td>-0.31</td>
<td>-0.42</td>
<td>-0.46</td>
<td>-0.47</td>
</tr>
<tr>
<td>Quaterly</td>
<td>-0.40</td>
<td>-0.44</td>
<td>-0.23</td>
<td>-0.52</td>
<td>-0.31</td>
<td>-0.52</td>
</tr>
<tr>
<td>L-Moments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weekly</td>
<td>-0.09</td>
<td>-0.16</td>
<td>-0.12</td>
<td>-0.14</td>
<td>-0.14</td>
<td>-0.13</td>
</tr>
<tr>
<td>Monthly</td>
<td>0.10</td>
<td>0.02</td>
<td>0.03</td>
<td>0.13</td>
<td>0.05</td>
<td>0.09</td>
</tr>
<tr>
<td>Quaterly</td>
<td>0.42</td>
<td>0.52</td>
<td>0.46</td>
<td>0.4</td>
<td>0.4</td>
<td>0.06</td>
</tr>
<tr>
<td>LD-Moments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weekly</td>
<td>-0.08</td>
<td>-0.33</td>
<td>-0.35</td>
<td>-0.19</td>
<td>-0.31</td>
<td>-0.27</td>
</tr>
<tr>
<td>Monthly</td>
<td>-0.14</td>
<td>0.16</td>
<td>-0.06</td>
<td>-0.05</td>
<td>-0.04</td>
<td>-0.24</td>
</tr>
<tr>
<td>Quaterly</td>
<td>-0.61</td>
<td>-0.78</td>
<td>-0.34</td>
<td>-0.28</td>
<td>-0.53</td>
<td>-0.13</td>
</tr>
</tbody>
</table>

Source: Euronext, 30" sampled intraday CAC40 French stock index quotes on the period 03-01-1995/02-28-2005. Computations by the authors.

To get a measure of the rate of decrease of the distribution function in its tail, an alternative method is to use, not a sample of the maxima, but Peaks-over-Threshold (POT Method), that is values of the random variable that exceed a cut-off point. In that case, the asymptotic distribution is the Generalized Pareto Distribution (GPD) that reads for an exceedance volatility \( \sigma \) (on the support \( IR^+ \) when \( \xi \geq 0 \) and \( \sigma \in [0, -\xi/\alpha] \) if \( \xi < 0 \)): 

\[ \sigma^2 = \left( \frac{\alpha}{\xi} \right)^2 \left( (1 - 2\xi) - \Gamma^2(1 - \xi) \right) \]

\[ \lambda_2 = \alpha \left( 1 - 2^{-\xi} \right) \frac{\Gamma(1 - \xi)}{\xi} \]

\[ \xi_2 = \frac{\alpha}{8} \left( 2Q_0(0.7) - 2Q_0(0.29) + Q_3(0.87) \right) \]

\[ \hat{z} = 2\hat{z}_3^{-1} - \ln(2)/\ln(3) \]

\[ Q_0(u) = -\frac{1 - (-\log(u))^{-\xi}}{\xi} \]
\[
G_\xi(\sigma) = \begin{cases} 
1 - \left(1 + \frac{\xi(h + \sigma)}{\alpha}\right)^{-\frac{1}{\xi}} & \text{if } \xi \neq 0 \\
1 - e^{-(h+\sigma)/\alpha} & \text{otherwise}
\end{cases}
\]

where \(\xi\) is also called the tail index, with \(\alpha\) and \(h\) scale and location parameters. Most of the common continuous distributions of statistics belong in this class of distributions. For example, the case where the tail index is positive corresponds to heavy-tailed distributions such as the Student’s t for example. The case where the tail index is null corresponds to distributions alike the normal or the lognormal whose tails decay exponentially. The short-tailed distributions with a finite endpoint, such as the uniform or beta, correspond to the case where the tail index is negative.

An important issue in implementing the POT method is the cut-off to be chosen. For example, if we are interested in the 5% quantile, in VaR context for example, then the chosen threshold must be larger enough around for being able to estimate the interesting quantile, but not too large since otherwise occurrences truly belonging to the centre of the distribution might contaminate the estimation of extremes. Indeed, the choice of the optimal threshold is a delicate issue since we are confronted with a bias-variance trade-off. If we choose a too low threshold, we might recover biased estimates because the limit theorem for the GEV distribution do not apply any more, while high thresholds generate estimates with high standard errors due to the limited number of observations. We choose to follow Neftçi (2000) and Bekiros and Georgoutsos (2005), fixing the cut-off value at a somewhat arbitrary reasonable level for our purpose and considering, for generating the exceedance data, the upper decile of distributions, meaning that it is the value under which 90% of the sample of volatilities lies. The GPD is estimated on the remaining extreme 10% of the distribution.\(^{11}\)

\(^{11}\) However, for the sake of confirmation, we also estimate the GPD tail indexes for the various measures using a threshold of 5%, with no resulting change in the qualitative results. One possible extension regarding this point is to estimate the threshold value to decide which extremes are really extremes, setting it at a level where the GPD still fits (see Gonzalo and Olmo, 2004).
Figure 7: Goodness-of-fit of the Generalized Pareto Distribution on Volatility Peaks-over-Threshold

Source: Euronext 3D resampled intraday CAC40 French stock index quotes on the period 03-01-1995/02-28-2005. Computations by the authors. Are represented in this figure, on the y-axis the empirical cumulative functions of the Realized Volatility (diamonds) and those of its Jump Component (triangles), altogether with their GPD best fits (thin lines) corresponding to estimated tail-indexes of -.16 (Realized Volatility) and -.19 (Jump Component) considering to the last decile of observed volatilities. Annualized Daily Volatilities are represented on the x-axis.

Like the GEV, the GPD can be estimated non-parametrically, by numerical maximisation of the log-likelihood or by matching the L-moments. In the latter case, an estimation of the tail index of the GPD denoted \( \hat{\xi} \) can be retrieved from the standardized third L-moment using the following relation (see Pandey et al., 2001):

\[
\hat{\xi} = -(3 \cdot \hat{\tau}_3 - 1)(\hat{\tau}_3 + 1)^{-1},
\]

where \( \hat{\tau}_3 \) is the empirical standardized third L-moment.

**Table 6: Summary of GPD Parameters Identification by Moment Matching**

<table>
<thead>
<tr>
<th>Order</th>
<th>Moments</th>
<th>L-Moments</th>
<th>TL-moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \mu = h + \frac{\alpha}{1 - \xi} )</td>
<td>( \lambda_1 = h + \frac{\alpha}{1 - \hat{\xi}} )</td>
<td>( \lambda_i = \frac{\alpha(5 - \xi)}{(2 - \xi)(3 - \xi)} )</td>
</tr>
<tr>
<td>2</td>
<td>( \sigma^2 = \frac{\alpha^2}{(1 + \xi^2)(1 - 2\xi)} )</td>
<td>( \lambda_2 = \frac{\alpha}{(1 + \xi)(2 + \xi)} )</td>
<td>( \lambda_2 = \frac{6\alpha}{(\xi + 2)(\xi + 3)(\xi + 4)} )</td>
</tr>
<tr>
<td>3</td>
<td>( s = 2(1 - \xi)(1 + \xi)^{1/2} ) ( (1 + 3\xi) )</td>
<td>( \tau_3 = \frac{(1 + \xi)}{(3 - \xi)} )</td>
<td>( \tau_3 = \frac{10(1 + \xi)}{9(5 - \xi)} )</td>
</tr>
</tbody>
</table>

Table 7 presents the results of the estimation of the GPD for extremes of various measures of volatility.
Table 7: Estimates of Tail Indexes of Generalized Pareto Distributions of Daily Log-volatilities via Moments Matching Method

<table>
<thead>
<tr>
<th></th>
<th>Realized</th>
<th>Jump Component</th>
<th>Close-to-close</th>
<th>Parkinson</th>
<th>Garman-Klass</th>
<th>Kunitomo</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Moments</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10% threshold</td>
<td>-0.15</td>
<td>-0.08</td>
<td>-0.19</td>
<td>-0.11</td>
<td>-0.12</td>
<td>-0.10</td>
</tr>
<tr>
<td>5% threshold</td>
<td>-0.15</td>
<td>-0.09</td>
<td>-0.20</td>
<td>-0.13</td>
<td>-0.15</td>
<td>-0.10</td>
</tr>
<tr>
<td>1% threshold</td>
<td>-0.18</td>
<td>-0.04</td>
<td>-0.26</td>
<td>-0.07</td>
<td>-0.13</td>
<td>-0.05</td>
</tr>
<tr>
<td><strong>L-Moments</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10% threshold</td>
<td>-0.16</td>
<td>-0.19</td>
<td>-0.32</td>
<td>-0.18</td>
<td>-0.19</td>
<td>-0.19</td>
</tr>
<tr>
<td>5% threshold</td>
<td>-0.24</td>
<td>-0.22</td>
<td>-0.36</td>
<td>0.37/</td>
<td>-0.27</td>
<td>-0.17</td>
</tr>
<tr>
<td>1% threshold</td>
<td>-0.33</td>
<td>-0.52</td>
<td>-0.53</td>
<td>-0.13</td>
<td>-0.48</td>
<td>-0.43</td>
</tr>
<tr>
<td><strong>TL-Moments</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10% threshold</td>
<td>-0.49</td>
<td>-0.49</td>
<td>-0.41</td>
<td>-0.50</td>
<td>-0.47</td>
<td>-0.48</td>
</tr>
<tr>
<td>5% threshold</td>
<td>-0.57</td>
<td>-0.49</td>
<td>-0.48</td>
<td>-0.53</td>
<td>-0.51</td>
<td>-0.53</td>
</tr>
<tr>
<td>1% threshold</td>
<td>-0.59</td>
<td>-0.49</td>
<td>-0.61</td>
<td>-0.59</td>
<td>-0.45</td>
<td>-0.47</td>
</tr>
</tbody>
</table>

Source: Euronext; 30' resampled intraday CAC40 French stock index quotes on the period 03-01-1995/02-28-2005. Computations by the authors. Are reported in this table the empirical Shape parameters for the various volatility measures, estimated by the moment matching Method with the last decile of observations.

Here again, all estimated tail indexes are negative, meaning that the log-volatility is not fat-tailed on the right side.

For lastly validating these results, we introduce hereafter a simple reality check: given the sample estimates of the parameters, it is now possible to compute the probability of observing the historical volatility peaks under the various measures and hypotheses. The sample spans from January 1995 to March 2005 and the highest daily volatility within these ten years unsurprisingly occurs on the 11th September 2001, whatever the measure. Using the tail index from the GPD, we now compute the probability of this event and its associated return time. Table 6 presents these probabilities.

Table 6: Estimated Probabilities of Observing the Highest Daily Volatility in the Sample

<table>
<thead>
<tr>
<th></th>
<th>Realized</th>
<th>Jump component</th>
<th>Close-to-close</th>
<th>Parkinson</th>
<th>Garman-Klass</th>
<th>Kunitomo</th>
<th>Rogers-Satchell</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conditional Probability</strong></td>
<td>0.18%</td>
<td>0.27%</td>
<td>0.34%</td>
<td>0.13%</td>
<td>0.35%</td>
<td>0.03%</td>
<td>0.08%</td>
</tr>
<tr>
<td><strong>Return-time (in years)</strong></td>
<td>21.80</td>
<td>28.90</td>
<td>11.90</td>
<td>30.80</td>
<td>11.40</td>
<td>115.80</td>
<td>47.50</td>
</tr>
</tbody>
</table>

Source: Euronext; 30' resampled intraday CAC40 French stock index quotes on the period 03-01-1995/02-28-2005. Computations by the authors. The highest volatility in the sample is observed on the 11/9, whatever the measure. The first line presents the conditional probability associated with the observation, given that it belongs to the last decile. The return time, in years, corresponds to the unconditional probability and assumes the decile measure to be accurate.

Though this reality check has a very limited statistical significance (return times being all longer that the sample size), it allows us to filter the results according to our subjective estimation of the likelihood of a major event of the magnitude of those of 9/11. According to the Realized Volatility measure, the corresponding shock on the market is relatively unlikely but not altogether incompatible with our sample. The Kunitomo’s and Rogers-Satchell’s, on the other hand, seem to underestimate the extreme probability whereas the simple close-to-close estimator might overestimate it.
From these estimations of the extreme values – backed-out using several methods and for various volatility estimators, we can prudently infer with a reasonable confidence that it is unlikely that a fat-tailed distribution is needed to fit the high volatilities. In particular, the Realized and Parkinson’s Extreme Value Volatilities are not incompatible with the log-normal hypothesis and thus the standard approximation is not significantly flawed.

5. Conclusion

The Realized Volatility, despite its known shortcomings, remains a benchmark to which measures of risk should be compared. We have shown here shown that, among the low-frequency measure, the instantaneous Parkinson’s volatility was the one approaching the closest the high-frequency measure. It thus should be the one retained when trying to get long horizon historical estimates, or to complement series of realized volatilities.

Estimations of the whole distribution of the empirical volatilities can not easily distinguish between the candidate functional forms. Given the rationale for estimating these distributions - retrieving possible risk - and the main differences between them - in the tails - it is natural to try instead to use Extreme Value Theory and concentrate on estimating the asymptotic distribution for the extreme measures of risk. The actual estimation, using different methods, for both the Generalized Extreme Value and the Generalized Pareto Distributions indicates that heavy-tailed distributions are not needed to fit the sample volatilities. A log-normal process, as in the traditional stochastic volatility model, seems able to reproduce the extreme empirical volatilities observed in our ten year ultra high frequency sample. As expected, the jump component of the realized volatility is mainly responsible for the shape of the distribution of the measure, since it represents all the peaks.

A final reality check shows the inadequacy of some of the simplest measures of risk, since they either underestimate or overestimate greatly the return times for the peak events observed in the sample. The use of the tail index estimates found for either the Parkinson’s or the Realized Volatility (when high frequency data is available) produces more reasonable-looking values for the return times of the high watermarks of volatility.

A practical application of these results will be to plug the appropriate estimates and distributions in the Index of Market Shocks (IMS, see Maillet and Michel, 2003 and 2005) in order to get a more accurate ranking of the historical crises and a precise estimation of the return times of extreme scenarii.

References


