Optimal maintenance decisions for berm breakwaters

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Abstract

To prevent coastal lines of defence from being affected by severe hydraulic loadings from the sea, berm breakwaters can be used. Although berm breakwaters are dynamically stable in the sense that they allow for some rock displacement, they can fail due to severe longshore rock transport. To avoid this type of failure, berm breakwaters have to be inspected and, if necessary, have to be repaired. A decision model is presented enabling cost-optimal maintenance decisions to be determined while taking account of the (possibly large) uncertainties in: (i) the limiting average rate of occurrence of breaches in the armour layer and (ii), given a breach has occurred, the limiting average rate of longshore rock transport. The stochastic process of rock displacement is modelled by a modified generalised gamma process, enabling us to explicitly take account of the uncertainty in these limiting averages.

Keywords

maintenance, gamma processes, berm breakwaters, decision theory, renewal theory.

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1 Introduction

In order to protect coastal lines of defence from being damaged by severe hydraulic loadings from the sea, berm breakwaters can be used (see e.g. Van der Meer [11] Van der Meer & Veldman [12] and Vrijling & Van Gelder [17]). The characteristic feature of berm breakwaters is that the original dynamically stable profile becomes statically stable under certain wave conditions. However, oblique wave attack can initiate longshore transport of stones along the center line of the berm breakwater. To avoid failure due to severe longshore rock transport, berm breakwaters have to be inspected and, if necessary, have to be repaired. The purpose of this paper is to develop a decision model for obtaining optimal inspection intervals whose expected maintenance costs are minimal.

Although most maintenance optimisation models are based on lifetime distributions or Markovian deterioration models, it is often hard to gather data for estimating their parameters. Moreover, in case of well-planned preventive maintenance, complete lifetimes will be observed rarely. In practice, there is often only (subjective) information available on limiting average rates of deterioration: for berm breakwaters, the stochastic deterioration process can be characterised by (i) the limiting average rate of occurrence of breaches in the armour layer and (ii) given a breach has occurred, the limiting average rate of longshore rock transport. In order that the processes of the occurrence of breaches and of longshore rock transport be based on their limiting average rates, they have been regarded as a mixture of geometrics and a scale mixture of gamma’s (a generalised gamma process), respectively.

According to Van Noortwijk & Peerbolte [16], three cost-based criteria can be used to compare maintenance decisions over an unbounded time-horizon: (i) the expected average costs per unit time, (ii) the expected discounted costs over an unbounded horizon, and (iii) the expected equivalent average costs per unit time. These costs can be determined with the aid of renewal theory, where the renewals are either preventive repairs (before failure) or corrective repairs (after failure).

The maintenance decision model which is proposed extends the model of Vrijling & Van Gelder [17] in the sense that the renewals not only take place at fixed preventive repair intervals but also upon failure, the costs are determined with respect to an unbounded horizon, and the dependence between the probability of preventive repair and the expected repair costs is taken into account. The parameters of the uncertainty distributions of the above two limiting averages are assessed using the simulation results of Vrijling & Van Gelder [17].

In The Netherlands, generalised gamma processes have also been used to model decision problems for optimising maintenance of the sea-bed protection of the Eastern-Scheldt barrier, beaches, and dykes (see Van Noortwijk et al. [13] and Speijker et al. [10]). Note that the developed decision model can be viewed as a delay-time model as studied by Christer & Waller [11]. In fact, the time lapse from an armour
breach until the time of failure due to longshore rock transport can be interpreted as the discrete delay time of a failure. Therefore, generalised gamma processes can be employed to assess delay-time distributions based on the physics of deterioration.

The paper is organised as follows. A brief description on berm breakwaters is given in Sec. 2. In Sec. 3 we model the event of failure due to severe longshore rock transport. We present the maintenance decision model for minimising the costs of inspection, repair, and failure in Sec. 4. Some necessary definitions and theorems are presented in two appendices.

## 2 Berm breakwaters

The main function of breakwaters is to prevent coastal lines of defence (e.g., sand dunes or cliffs) from being affected by severe hydraulic loadings from the sea. Recently, the attention has been shifted from statically stable rubble-mound breakwaters to dynamically stable berm breakwaters (see e.g., Van der Meer [11], Van der Meer & Veldman [12], and Vrijling & Van Gelder [17]). The profiles of statically stable structures are not permitted to change under severe wave conditions whereas the profiles of dynamically stable structures (such as berm breakwaters and beaches of sand, gravel, shingle, and rock) may change according to the wave climate.

The main components of a berm breakwater are the core (with stones of diameter 0.5 m) and the armour layer (with stones of diameter 0.8 m) (see Fig. 1). A berm breakwater is said to be dynamically stable when the net cross-shore transport of stones is zero and its profile has reached an equilibrium under certain wave conditions. In fact, the originally built profile becomes dynamically stable when wave attack moves rock in the berm partly upward to the crest and partly downward to the toe and the sand; thus reshaping the seaward slope into a (more) statically stable S-shape profile (see Fig. 1).

![Figure 1: The cross-section of a berm breakwater: the originally built (dynamically stable) profile and the S-shape (statically stable) profile. S.W.L. means still-water level: the surface of the water if all wave and wind action were to cease.](image)

A berm breakwater can fail due to longshore transport of the armour elements which occurs when oblique wave attack results in wave forces parallel to the alignment.
of the structure. We define a failure as the event at which the stones of the armour layer have been displaced to such a degree that the core is unstable and needs to be reconstructed. To reveal possible longshore transport of stones berm breakwaters have to be inspected and if necessary have to be repaired. In this paper we develop a decision model that enables cost-optimal maintenance decisions to be determined while taking account of the main uncertainties in the stochastic process of rock displacement.

3 The stochastic process of rock displacement

In modelling the maintenance of berm breakwaters on the basis of the stochastic process of rock displacement there are mainly two uncertainties involved (see also Fig. 2): (i) the probability of an initial breach of the armour layer and (ii) given a breach has occurred the limiting average rate of longshore rock transport. In fact an armour breach initiates longshore rock transport. Next we study these two deterioration characteristics which are judged to be independent in more detail.

3.1 The stochastic process of longshore rock transport

In this subsection we derive a probabilistic model for the process of longshore rock transport based on the unknown limiting average rate. Let us denote the non-decreasing stochastic process of longshore rock transport by \{X(t) : t ≥ 0\} where X(t) represents the cumulative amount of transported rock at time t and X(0) = 0 with probability one. For every uniform time-partition in time-intervals of length τ > 0 we write

\[ D_i(τ) = X(iτ) - X((i - 1)τ) \] i ∈ \mathbb{N}. Furthermore due to the lack of data we judge the infinite sequence of increments \{D_i(τ) : i ∈ \mathbb{N}\} to be exchangeable i.e. the order in which the increments occur is irrelevant. In mathematical terms this can be interpreted as that the probability density function of the random vector \( (D_1(τ), \ldots, D_n(τ)) \) is invariant under all \( n! \) permutations of the coordinates i.e.

\[ p_{D_1(τ), \ldots, D_n(τ)}(δ_1, \ldots, δ_n) = p_{D_1(τ), \ldots, D_n(τ)}(δ_{π(1)}, \ldots, δ_{π(n)}) \] \tag{1}

where \( π \) is any permutation of 1, \ldots, n for all \( n ∈ \mathbb{N} \) and \( τ > 0 \).

In order that a stochastic deterioration process with non-negative exchangeable increments be based on the unknown limiting average rate Van Noortwijk, Cooke & Misiewicz [14] have argued that it can best be regarded as a generalized gamma process. For this process the joint probability density function of the increments \( D_1(τ), \ldots, D_n(τ) \) is given by a mixture of conditionally independent gamma densities:

\[ p_{D_1(τ), \ldots, D_n(τ)}(δ_1, \ldots, δ_n) = \int_0^∞ \prod_{i=1}^n \frac{a^{aτi-1}}{Γ(aτ)} \left[ \frac{aτ}{θ} \right]^{aτ} \exp \left\{ -\frac{aτδ_i}{θ} \right\} dP_θ(τ)(θ) \] \tag{2}
for some constant $a > 0$ with

$$E(X(n\tau)) = E(n\Theta(\tau)), $$

$$\text{Var}(X(n\tau)) = \left[1 + \frac{1}{n a \tau}\right] E([n\Theta(\tau)]^2) - [E(n\Theta(\tau))]^2$$

(3)

for all $\tau > 0 \Gamma$ provided the first and the second moment of the probability distribution of $\Theta(\tau)$ exist. By the strong law of large numbers for exchangeable random quantities $\Gamma$ the probability distribution $P_{\Theta(\tau)}$ on the random quantity $\Theta(\tau)$ represents the uncertainty in the unknown limiting average amount of longshore rock transport per time-interval of length $\tau$: $\lim_{n \to \infty} [(\sum_{i=1}^n D_i(\tau))/n]$.

A useful property of the generalised gamma process is that the mixture of gamma’s in Eq. (2) transforms into a mixture of exponentials if $\tau = a^{-1}$:

$$p_{D_1(a^{-1}), \ldots, D_n(a^{-1})} (\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\theta} \exp \left\{ -\frac{\delta_i}{\theta} \right\} dP_\Theta (\theta) = f_n (\sum_{i=1}^n \delta_i)$$

(4)

for $(\delta_1, \ldots, \delta_n) \in \mathbb{R}_+^n$ and zero otherwise $\Gamma$ where $\mathbb{R}_+ = [0, \infty)$. The infinite sequence of random quantities $\{D_i(a^{-1}) : i \in \mathbb{N}\}$ is said to be $l_1$-isotropic (or $l_1$-norm symmetric) $\Gamma$ since its distribution can be written as a function of the $l_1$-norm. The unit time for which the increments of longshore rock transport are $l_1$-isotropic can be obtained among others by specifying the variance of the generalised gamma process in Eq. (3). For fixed $\tau > 0 \Gamma$ the smaller the unit-time length for which the increments are $l_1$-isotropic $\Gamma$, i.e. the smaller $\Delta = a^{-1} \Gamma$ the more deterministic the deterioration process. As we shall see in Sec. 4 $\Gamma$ for this unit-time length $\Gamma$ denoted by $\Delta = a^{-1} \Gamma$ many probabilistic properties of the stochastic process $\Gamma$ like the probability of exceedence of a failure level $\Gamma$ can be expressed in explicit form conditional on the limiting average.

In conclusion $\Gamma$ we advocate regarding the stochastic process of longshore rock transport as a generalised gamma process with probability distribution on the limiting average rate of longshore rock transport. To keep the mathematics of the decision model tractable $\Gamma$ we impose the property of posterior linearity introduced by Diaconis & Ylvisaker [5] $\Gamma$ i.e. $E(X(2\tau) | D_1(\tau) = \delta_1) = c_1 \delta_1 + c_2$ for some constants $c_1, c_2 > 0$ and $\tau > 0$. If this property holds $\Gamma$ then the mixing distribution in Eq. (2) is an inverted gamma distribution.

From now on $\Gamma$ we consider increments of longshore rock transport that are $l_1$-isotropic with respect to the units of time $\{([i \Delta, i \Delta) : i \in \mathbb{N}\}$. For notational convenience $\Gamma$ let $D_i = D_i(\Delta) \Gamma X_n = \sum_{i=1}^n D_i$ for all $n \in \mathbb{N}$ $\Gamma$ and $\Theta$ represent the uncertainty in the limiting average rate of longshore rock transport $\Gamma \lim_{n \to \infty} [(\sum_{i=1}^n D_i)/n] \Gamma$ with probability density function $Ig(\theta | \nu, \mu)$ (for the definition of the inverted gamma distribution $\Gamma$ see the appendix).
3.2 The stochastic process of armour breaches

Beside the uncertainty in the limiting average rate of longshore rock transport the process of the occurrence of breaches in the armour layer must be specified. Let us denote the times at which armour breaches occur by the infinite sequence of non-negative discrete random quantities $T_1, T_2, \ldots$ (which are defined with respect to units of time of length $\Delta$). Since rocks start to move due to external causes (i.e. severe waves) at a rate that does not change in time we may judge the random quantities $T_1, T_2, \ldots$ to be exchangeable and to exhibit the “lack of memory” property. The “lack of memory” property means that the discrete probability function of the remaining time until the occurrence of the first armour breach does not depend on the fact that no breach has occurred yet.

Under the above two assumptions we can write the joint probability function of $T_1, \ldots, T_n$ as a mixture of conditionally independent geometric distributions (see Diaconis & Freedman [4]):

$$\Pr \{ T_1 = t_1, \ldots, T_n = t_n \} = \int_0^1 \prod_{i=1}^n \phi(1 - \phi)^{t_i} \, d\Pr \{ \Phi \leq \phi \}$$  \hspace{1cm} (5)

for $t_i = 0, 1, 2, \ldots \Gamma i = 1, \ldots, n\Gamma$ and zero otherwise. The conditional mean can be written as $E(T_i|\phi) = \phi^{-1} - 1\Gamma i = 1, \ldots, n$. By the strong law of large numbers for exchangeable random quantities the random quantity $\Phi$ may be interpreted as the limiting relative frequency of the times at which initial armour breaches occur per unit time of length $\Delta\Gamma$ i.e. $\lim_{n \to \infty}[n/(n + \sum_{i=1}^n T_i)]$. If in addition the property of posterior linearity holds then the mixing distribution in Eq. (5) is a beta distribution say $Be(\phi|a, b)$ (see Diaconis & Ylvisaker [5]).

4 Optimal maintenance decisions

4.1 The maintenance model

To obtain optimal maintenance decisions under uncertainty we can use statistical decision theory (see DeGroot [3] Ch. 8]). Let a berm breakwater be inspected at times \{ $jk\Delta : j \in \mathbb{N}$ \} for fixed $k \in \mathbb{N}$. Let $L(\phi, \theta, k)$ be the (monetary) loss when the decision-maker chooses inspection interval $k$ the limiting relative frequency of armour breaches is $\phi\Gamma$ and the limiting average rate of longshore rock transport is $\theta$. The decision-maker can best choose an inspection interval $k^*$ whose expected loss is minimal. A decision $k^*$ is called an optimal decision when

$$E(\{ L(\Phi, \Theta, k^*) \}) = \min_{k \in 1,2,\ldots} E(\{ L(\Phi, \Theta, k) \}).$$  \hspace{1cm} (6)

The resistance of the berm breakwater denoted by $R\Gamma$ is defined as the number of stones belonging to the armour layer where $r_0$ is the resistance of the originally
built profile. When an initial breach of the armour layer occurs at time \((i - 1)\Delta\Gamma\) i.e. when \(T_i = i - 1\Gamma\) the longshore rock transport at the inspection time \(k\Delta\Gamma\) \(i \leq k\) is represented by the random quantity \(X_{k-i+1}\) where \(i, k = 1, 2, \ldots\) and \(X_n \sim \text{Ga}(n, 1/\theta)\) for all \(n \in \mathbb{N}\) (for the definition of the gamma distribution see the appendix). The expected longshore rock transport at time \(k\Delta\Gamma\) given \(\phi\) and \(\theta\Gamma\) can be obtained by conditioning on the possible values of \(T_1\) and using Eqs. (4)–(5) and (28):

\[
\sum_{i=1}^{k} \Pr\{ T_1 = i - 1 \mid \phi \} E(X_{k-i+1} \mid T_1 = i - 1, \theta) = \\
= \sum_{i=1}^{k} \phi(1 - \phi)^{i-1}[k - (i - 1)] \theta = \xi_{k,i}(\phi) \theta. \tag{7}
\]

Since repairs bring the berm breakwater into the “as good as new state” we may regard the maintenance process as a renewal process (see Fig. 2). Each renewal cycle ends either upon a failure or at an inspection time \(jk\Delta\) when the inspection reveals that a preventive repair should be carried out (for some \(j \in \mathbb{N}\)). A failure is defined as the event in which the resistance \(R\) drops below the failure level \(s\): \(R < s\). A preventive repair is defined as the event at which inspection reveals that longshore rock transport has taken place but no failure has occurred: \(s \leq R < r_0\). Inspection takes a negligible amount of time does not degrade the berm breakwater and entails fixed costs \(c_f\). The costs of failure are \(c_F\) (costs of reconstructing the core and of possible damage to the coastline) while the costs of repair consist of the fixed costs \(c_R\) (costs of mobilisation) and the variable costs \(c_V\) (costs per rock). Let the renewal times be conditionally independent random quantities having a discrete probability function \(p_i(\phi, \theta, k)\) \(i \in \mathbb{N}\) when the limiting average rates are \(\phi\) and \(\theta\Gamma\) and the decision-maker chooses inspection decision \(k\). The costs associated with a renewal at time \(i\Delta\) are denoted by \(c_i(\phi, \theta, k)\) \(i \in \mathbb{N}\).

Since berm breakwaters are planned to function for a very long time maintenance decisions can best be compared over an unbounded time-horizon. According to Van Noortwijk & Peerbolte [16] there are basically three cost-based criteria that can serve as loss functions in Eq. (6): (i) the expected average costs per unit time (ii) the expected discounted costs over an unbounded horizon and (iii) the expected equivalent average costs per unit time. These cost-based criteria can be obtained using the discrete renewal theorem (see Feller [6] Ch. 13 and Karlin & Taylor [8] Ch. 3).

First the expected average costs per unit time are determined by averaging the expected costs over an unbounded horizon:

\[
L(\phi, \theta, k) = \lim_{n \to \infty} \frac{C(n, \phi, \theta, k)}{n} = \frac{\sum_{i=1}^{\infty} c_i(\phi, \theta, k)p_i(\phi, \theta, k)}{\sum_{i=1}^{\infty} ip_i(\phi, \theta, k)}, \tag{8}
\]

where \(C(n, \theta, k)\) are the expected costs in time-interval \((0, n\Delta]\). Eq. (8) is a well-known result from renewal reward theory (see e.g. Ross [9]).

Second the expected discounted costs over an unbounded horizon are determined by summing the expected discounted values of the costs over an unbounded horizon.
failure
resistance
[breaches of armour layer]
prevention repair

Figure 2: The deterioration process of a berm breakwater regarded as a renewal process: each renewal cycle ends either upon an inspection (I), revealing that a preventive repair should be carried out, or upon a failure (F). The inspection interval is taken to be $k = 4$.

where the discounted value of the costs $c_n$ in unit time $n$ is defined to be $\alpha^n c_n$ with discount factor $\alpha = [1 + (r/100)]^{-1}$ and discount rate $r \% \ (r > 0)$:

$$L_\alpha (\phi, \theta, k) = \lim_{n \to \infty} C_\alpha(n, \phi, \theta, k) = \frac{\sum_{i=1}^{\infty} \alpha^i c_i(\phi, \theta, k) p_i(\phi, \theta, k)}{1 - \sum_{i=1}^{\infty} \alpha^i p_i(\phi, \theta, k)}, \quad (9)$$

where $C_\alpha(n, \phi, \theta, k)$ are the expected discounted costs in time-interval $(0, n\Delta)$.

Third, the expected equivalent average costs per unit time are determined by averaging the discounted costs. The notion of equivalent average costs relates the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time as $\alpha$ tends to 1 from below:

$$\lim_{\alpha \to 1} (1 - \alpha)L_\alpha(\phi, \theta, k) = L(\phi, \theta, k). \quad (10)$$

Before deriving the above cost-based criteria, we need to express the failure probability of a berm breakwater in terms of the limiting averages $\phi$ and $\theta$. By summing over the possible values of the first time at which an armour breach occurs $T_i \Gamma$ the probability of failure due to longshore rock transport in unit time $i$ becomes

$$v_i(\phi, \theta) = \Pr \{ \text{failure in } ([i - 1]\Delta, i\Delta] \mid \phi, \theta \} \bigg| \sum_{h=1}^{i} \Pr \{ T_i = i - h \mid \phi \} \Pr \{ X_{h-1} \leq y, X_h > y \mid \theta \}$$

(11)
for \( i = 1, 2, \ldots \) and \( y = r_0 - s \Gamma \) where

\[
\Pr \{ X_{h-1} \leq y, X_h > y \mid \theta \} = \frac{1}{(h-1)!} \left[ \frac{y}{\theta} \right]^{h-1} \exp \left\{ -\frac{y}{\theta} \right\} = q_h(\theta), \tag{12}\]

\( h = 1, 2, \ldots \) is a Poisson distribution (see Van Noortwijk \& Cooke \& Kok [13]). Using Eq. (5) the probability of failure in unit time \( i \) can be rewritten as a recursive formula in the following way:

\[
v_i(\phi, \theta) = \sum_{h=1}^{i} \phi (1 - \phi)^{i-h} q_h(\theta) = (1 - \phi) v_{i-1}(\phi, \theta) + \phi q_i(\theta), \tag{13}\]

\( i = 1, 2, \ldots \) and \( v_0(\phi, \theta) = 0 \Gamma \) with mean lifetime and variance given by

\[
E(T_1 \mid \phi) + E(H \mid \theta) = (1 - \phi)/\phi + 1 + (y/\theta),
\]

\[
\text{Var}(T_1 \mid \phi) + \text{Var}(H \mid \theta) = (1 - \phi)/\phi^2 + (y/\theta), \tag{14}\]

where \( T_1 \) has a geometric distribution with parameter \( 0 < \phi < 1 \) and \( H \) has a Poisson distribution with parameter \( y/\theta > 0 \) (see Eqs. (5) and (12) respectively). By Eqs. (23-29) of Appendix B the expected average costs per unit time \( \Gamma \) are

\[
L(\phi, \theta, k) = \frac{c_l + c_p \left[ 1 - (1 - \phi)^k \right] + \sum_{i=1}^{k} [c_F - (c_l + c_p)] v_i(\phi, \theta)}{k + \sum_{i=1}^{k} [i - k] v_i(\phi, \theta)} + \frac{c_v \left[ \xi_{k,k}(\phi) - \sum_{h=1}^{k+1} \xi_{k-h+2,h}(\phi) q_h(\theta) \right]}{k + \sum_{i=1}^{k} [i - k] v_i(\phi, \theta)}, \tag{15}\]

whereas the expected discounted costs over an unbounded horizon \( \Gamma \) are

\[
L_\alpha(\phi, \theta, k) = \frac{\alpha^k \left\{ c_l + c_p \left[ 1 - (1 - \phi)^k \right] \right\} + \sum_{i=1}^{k} \left[ \alpha^i c_F - \alpha^k (c_l + c_p) \right] v_i(\phi, \theta)}{[1 - \alpha^k] - \sum_{i=1}^{k} [\alpha^i - \alpha^k] v_i(\phi, \theta)} + \frac{\alpha^k c_v \left[ \xi_{k,k}(\phi) - \sum_{h=1}^{k+1} \xi_{k-h+2,h}(\phi) q_h(\theta) \right]}{[1 - \alpha^k] - \sum_{i=1}^{k} [\alpha^i - \alpha^k] v_i(\phi, \theta)}. \tag{16}\]

In conclusion we recommend choosing an optimal inspection interval \( k^* \) for which the expected average costs per unit time \( \Gamma E(L(\Phi, \Theta, k^*)) \) for the expected equivalent average costs per unit time \( \Gamma E((1 - \alpha) L_\alpha(\Phi, \Theta, k^*)) \) are minimal. The choice for one or the other depends on the application and no general recommendation can be given. The expected costs are taken with respect to the probability distributions of the unknown random quantities \( \Phi \) and \( \Theta \) which are judged to be independent. As \( \alpha \) tends to 1 from below the expected equivalent average costs approach the expected average costs (by Lebesgue’s Theorem of Dominated Convergence) we may interchange the order of the operations of expectation and passing to the limit).
4.2 Numerical results

Next we apply the above maintenance model to the data obtained by Vrijling & Van Gelder [17] (see Table 1). We consider a hypothetical harbour (in India) which is protected by a berm breakwater and focus on one breakwater section with an armour layer having a rock volume of about 2500 stones. The costs of failure not only consist of the costs of reconstructing the berm breakwater but also of possible damage due to wave disturbance in the harbour basin and of resulting downtime in ship handling.

The unit time for which the increments of longshore rock transport are distributed as mixtures of exponentials \((\Delta = 1 \text{ year})\) and the parameters of the probability distributions on the limiting averages rates \(\Phi\) and \(\Theta\) have been assessed such that they fit the data of Vrijling & Van Gelder [17] (see Table 1). In doing so the probabilities of no armour breach per unit time i.e. \(E([1 - \Phi]^I)\) \(i = 1, \ldots, 50\) are given by Fig. 3 and the probability of failure in time-interval \((0, 50)\) has the value 0.22. When using the parameters of Table 1 and applying Monte Carlo integration (number of samples: 10,000) the average costs per year and the equivalent average costs per year are represented by the curves in Fig. 4. The optimal decision with respect to the criterion of average costs is \(k^* = 4\) with expected average costs per unit time of 6371 Dfl whereas the optimal decision with respect to the criterion of equivalent average costs is \(k^*_e = 5\) with expected equivalent average costs per unit time of 5676 Dfl.

In Table 2 the optimal inspection intervals are presented for different costs of failure (for a discussion on determining these costs of failure see Hauer et al. [7]): the higher the costs of failure the smaller the optimal inspection interval. Also we have investigated the sensitivity of the optimum to the variances of the probability distributions of the limiting averages rates \(\Phi\) and \(\Theta\) (while keeping the means unchanged). In Figs. 5 and 6 the expected (equivalent) average costs per unit time are shown for respectively \(\text{Var}(\Phi)\) and \(\text{Var}(\Theta)\) having a value one thousand times smaller than the ones in Table 1. It can be concluded that the set of (nearly) optimal decisions is more sensitive to the uncertainty in the limiting average rate of longshore rock transport than to the uncertainty in the limiting average rate of occurrence of breaches in the armour layer. All in all the larger the uncertainty in the stochastic process of rock displacement the smaller the optimal inspection interval.

5 Conclusions

In this paper we have presented a decision model which enables the decision-maker to optimise maintenance of berm breakwaters. The model has been derived on the basis of the probability distributions of (i) the limiting average rate of occurrence of breaches in the armour layer and (ii) given a breach has occurred the limiting average rate of longshore rock transport. As decision criteria we have used the expected average costs per unit time (no discounting) and the expected equivalent average costs per unit time.
Table 1: The parameters of the maintenance model for the berm breakwater.

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<th>description</th>
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<td>unit time</td>
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<td>year</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>average rate longshore rock transport</td>
<td>$(0, \infty)$</td>
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<td>$c_I$</td>
<td>costs of inspection</td>
<td>$10^3$</td>
<td>Dfl</td>
</tr>
<tr>
<td>$c_P$</td>
<td>fixed costs of preventive repair</td>
<td>$10^4$</td>
<td>Dfl</td>
</tr>
<tr>
<td>$c_V$</td>
<td>variable costs of preventive repair</td>
<td>$10^2$</td>
<td>Dfl/stone</td>
</tr>
<tr>
<td>$c_F$</td>
<td>costs of failure</td>
<td>$2.5 \times 10^6$</td>
<td>Dfl</td>
</tr>
<tr>
<td>$r_0$</td>
<td>initial resistance</td>
<td>2500</td>
<td>stones</td>
</tr>
<tr>
<td>$s$</td>
<td>failure level</td>
<td>0</td>
<td>stones</td>
</tr>
<tr>
<td>$k$</td>
<td>inspection-interval length</td>
<td>IN</td>
<td>unit time</td>
</tr>
</tbody>
</table>

Table 2: Optimal inspection intervals for different costs of failure.

<table>
<thead>
<tr>
<th>$c_F$</th>
<th>0.75</th>
<th>1.00</th>
<th>2.50</th>
<th>5.00</th>
<th>7.50</th>
<th>10.0</th>
<th>25.0</th>
<th>$\times 10^6$ Dfl</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k^*$</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>year</td>
</tr>
<tr>
<td>$k_<em>^</em>$</td>
<td>24</td>
<td>11</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td>year</td>
</tr>
</tbody>
</table>
time (discounting). The stochastic process of rock displacement has been regarded as a modified generalised gamma process.

The maintenance models that are presented in this paper have the following advantages: they enable optimal inspection intervals to be determined under uncertainty; they estimate how much money is needed for the future maintenance of berm breakwaters; and they can be expressed in explicit form when the limiting average rates of rock displacement are given.

Even though the decision model has been used for obtaining optimal inspection intervals only, it can also be applied for determining the optimal resistance of a berm breakwater (in terms of the number of stones). The decision model is a delay-time model. It can be applied to many fields of engineering to solve problems in maintenance optimisation and life cycle costing.
Figure 3: The probability of no breach in the armour layer per unit time: i.e. $E([1 - \Phi]^i), i = 1, \ldots, 50$, for $\Phi \sim \text{Be}(0.5, 0.5)$ with $E(\Phi) = 0.5$ and $\text{Var}(\Phi) = 1.25 \times 10^{-1}$.

Figure 4: The expected average costs per unit time and the expected equivalent average costs per unit time for $\Phi \sim \text{Be}(0.5, 0.5)$ and $\Theta \sim \text{Ig}(2.13, 55.36)$ with $E(\Phi) = 0.5$, $\text{Var}(\Phi) = 1.25 \times 10^{-1}$, $E(\Theta) = 49$, and $\text{Var}(\Theta) = 1.84 \times 10^{4}$. 

Figure 5: The expected average costs per unit time and the expected equivalent average costs per unit time for $\Phi \sim \text{Be}(1000, 1000)$ and $\Theta \sim \text{Ig}(2.13, 55.36)$ with $E(\Phi) = 0.5$, $\text{Var}(\Phi) = 1.25 \times 10^{-4}$, $E(\Theta) = 49$, and $\text{Var}(\Theta) = 1.84 \times 10^4$.

Figure 6: The expected average costs per unit time and the expected equivalent average costs per unit time for $\Phi \sim \text{Be}(0.5, 0.5)$ and $\Theta \sim \text{Ig}(132.2, 6428)$ with $E(\Phi) = 0.5$, $\text{Var}(\Phi) = 1.25 \times 10^{-1}$, $E(\Theta) = 49$, and $\text{Var}(\Theta) = 1.84 \times 10^1$. 

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A Appendix: Definitions and theorems

Definition 1 (Gamma distribution.) A random quantity $X$ has a gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$ if its probability density function is given by:

$$\text{Ga}(x \mid a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp\{-bx\} \text{I}_{(0,\infty)}(x).$$

Definition 2 (Inverted gamma distribution.) A random quantity $Y$ has an inverted gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$ if $X = Y^{-1} \sim \text{Ga}(a, b)$. Hence, the probability density function of $Y$ is:

$$\text{Ig}(y \mid a, b) = \frac{b^a}{\Gamma(a)} y^{-(a+1)} \exp\{-b/y\} \text{I}_{(0,\infty)}(y).$$

Definition 3 (Beta distribution.) A random quantity $X$ has a beta distribution with parameters $a, b > 0$ if its probability density function is given by:

$$\text{Be}(x \mid a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \text{I}_{[0,1]}(x).$$
**Theorem 1** Suppose the infinite sequence of random quantities \( \{D_i : i \in \mathbb{N}\} \) is \( l_1 \)-isotropic and \( X_n = \sum_{i=1}^{n} D_i \) for all \( n \in \mathbb{N} \), then

\[
E \left( [X_j]^m I_{[0,y]}(X_{n-1}) I_{(y, \infty)}(X_n) \right) = \frac{1}{(n-1)!} \left[ \frac{y}{\theta} \right]^{n-1} \exp \left\{ -\frac{y}{\theta} \right\},
\]

for \( j, n = 1, 2, \ldots, j < n, m \geq 0, y \in (0, \infty) \), where \( I_A(x) = 1 \) if \( x \in A \) and \( I_A(x) = 0 \) if \( x \notin A \). The quantity \((n)_m = \Gamma(n+m)/\Gamma(n)\) is known as Pochhammer’s symbol.

**Proof:**

Since \( X_n = \sum_{i=1}^{n} D_i \) for all \( n \in \mathbb{N} \) it follows that the integration bounds are determined by \( 0 \leq X_1 \leq \ldots \leq X_{n-1} \leq X_n \). Moreover \( X_{n-1} \leq y \) and \( X_n > y \Gamma \) and the Jacobian equals one. Hence we may write

\[
E \left( [X_j]^m I_{[0,y]}(X_{n-1}) I_{(y, \infty)}(X_n) \right) = \frac{1}{(n-1)!} \left[ \frac{y}{\theta} \right]^{n-1} \exp \left\{ -\frac{y}{\theta} \right\} dx_n \ldots dx_1.
\]

This multiple integral can be solved in four steps.

First integrating out the variable \( x_n \) gives:

\[
\int_{x_n=y}^{\infty} \left[ \frac{1}{\theta} \right]^{n} \exp \left\{ -\frac{x_n}{\theta} \right\} dx_n = \left[ \frac{1}{\theta} \right]^{n-1} \exp \left\{ -\frac{y}{\theta} \right\}.
\]

Second, using the Dirichlet integral entails:

\[
\int_{x_1=0}^{y} \cdots \int_{x_{j+1}=x_j}^{y} 1 dx_{j+1} \cdots dx_{j+(n-j-1)} = \frac{[y-x_j]^{n-j-1}}{(n-j-1)!}.
\]

Third, reversing the order of integration and applying the beta integral leads to:

\[
\int_{x_1=0}^{y} \cdots \int_{x_{j+1}=x_j}^{y} x_j^m [y-x_j]^{n-j-1} dx_j \cdots dx_2 dx_1 = \int_{x_1=0}^{y} x_j^m dy \int_{x_1=0}^{y} x_j^m [y-x_j]^{n-j-1} dx_1 dx_j \cdots dx_{j-1} dx_j = \frac{1}{(j-1)!} \int_{x_1=0}^{y} x_j^{m+j-1} dx_j = \frac{1}{(j-1)!} y^{m+n-1} B(m+j,n-j).
\]

Finally, combining Eqs. (18-20) proves the theorem. \( \Box \)
B Appendix: The expected maintenance costs

In order to compare maintenance decisions over unbounded horizons for berm breakwaters, we need to determine two cost-based criteria: (i) the expected average costs per unit time Eq. (8) and (ii) the expected discounted costs over an unbounded horizon Eq. (9). These costs can be computed using renewal theory where the renewals are the maintenance actions restoring a berm breakwater to its “originally built” profile. Defining a renewal cycle as the time-period between two renewals, we can derive explicit expressions for the expected cycle costs, the expected cycle length, the expected discounted cycle costs, and the expected discounted cycle length. Recall that inspections are scheduled at times \( jk \Delta : j \in \mathbb{N} \) with inspection interval \( k \in \mathbb{N} \). The costs of inspection are \( c_I \), the costs of failure are \( c_F \), the fixed costs of preventive repair are \( c_P \) and the variable costs of preventive repair are \( c_V \). For convenience, let \( \Delta = 1 \) and \( c_V = 0 \); the case \( c_V > 0 \) is considered in the last subsection.

The probabilities of failure and preventive repair per inspection interval.
By using Eqs. (11-13) and reversing the order of summation, the conditional probability of failure in time-interval \((0, k]\) given \( \phi \) and \( \theta \) has the following forms

\[
\Pr \{ \text{failure in } (0, k] \mid \phi, \theta \} = \sum_{i=1}^{k} v_i(\phi, \theta) = \sum_{h=1}^{k} \left[ 1 - (1 - \phi)^{k-h+1} \right] q_h(\theta) = \sum_{i=1}^{k} \phi (1 - \phi)^{i-1} \sum_{h=1}^{k-i+1} q_h(\theta).
\]

Similarly, the conditional probability of preventive repair in time-interval \((0, k]\) given \( \phi \) and \( \theta \) can be written as

\[
\Pr \{ \text{preventive repair at } k \mid \phi, \theta \} = \sum_{i=1}^{k} \phi (1 - \phi)^{i-1} \left[ 1 - \sum_{h=1}^{k-i+1} q_h(\theta) \right] = [1 - (1 - \phi)^k] - \sum_{i=1}^{k} v_i(\phi, \theta).
\]

The expected cycle costs \((c_V = 0)\).
On the basis of Eqs. (21-22), the expected cycle costs can be decomposed into the expected cycle costs due to inspection, preventive repair \((c_V = 0)\) and failure:

\[
\sum_{i=1}^{\infty} c_i(\phi, \theta, k)p_i(\phi, \theta, k) = \sum_{j=1}^{\infty} [jc_I + c_P] \Pr \{ \text{preventive repair at } jk \mid \phi, \theta \} +
\]
where the second step follows from the “lack of memory” property of the geometric distribution (in Eq. (5)).

The expected cycle length.
Since each renewal cycle ends either upon a failure or at a preventive repair it the expected cycle length can be written as

\[
\sum_{i=1}^{\infty} i p_i(\phi, \theta, k) = \sum_{j=1}^{\infty} j k \Pr \{ \text{preventive repair at } jk \mid \phi, \theta \} +
\]

\[
\sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} n \Pr \{ \text{failure in } (n-1, n) \mid \phi, \theta \}
\]

\[
= \sum_{j=1}^{\infty} jk (1 - \phi)^{(j-1)k} \Pr \{ \text{preventive repair at } k \mid \phi, \theta \} +
\]

\[
\sum_{j=1}^{\infty} \sum_{i=1}^{k} [(j-1)k + i] (1 - \phi)^{(j-1)k} \Pr \{ \text{failure in } (i-1, i) \mid \phi, \theta \}
\]

\[
= \sum_{j=1}^{\infty} jk (1 - \phi)^{(j-1)k} \left\{ [1 - (1 - \phi)^{k}] - \sum_{i=1}^{k} v_i(\phi, \theta) \right\} +
\]

\[
\left( 1 - (1 - \phi)^{k} \right) \]
The expected discounted cycle costs consist of the expected discounted costs due to inspection, preventive repair at \( j \) and failure:

\[
\sum_{i=1}^{\infty} \alpha^i c_i(\phi, \theta, k) P_i(\phi, \theta, k) = \\
= \sum_{j=1}^{\infty} \left[ \left( \sum_{h=1}^{j} \alpha^{hk} \right) c_I + \alpha^{jk} c_P \right] \Pr \{ \text{preventive repair at } jk | \phi, \theta \} + \\
\sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \left[ \left( \sum_{h=1}^{j-1} \alpha^{hk} \right) c_I + \alpha^{(j-1)k+i} c_F \right] (1 - \phi)^{(j-1)k} \Pr \{ \text{failure in } (n-1, n] | \phi, \theta \} \\
= \sum_{j=1}^{\infty} \left[ \left( \sum_{h=1}^{j} \alpha^{hk} \right) c_I + \alpha^{jk} c_P \right] (1 - \phi)^{(j-1)k} \left\{ [1 - (1 - \phi)^k] - \sum_{i=1}^{k} v_i(\phi, \theta) \right\} + \\
\sum_{j=1}^{\infty} \left( \sum_{h=1}^{j-1} \alpha^{hk} \right) (1 - \phi)^{(j-1)k} c_I \sum_{i=1}^{k} v_i(\phi, \theta) + \sum_{j=1}^{\infty} \left[ \alpha (1 - \phi) \right]^{(j-1)k} c_F \sum_{i=1}^{k} \alpha^i v_i(\phi, \theta) \\
= \sum_{j=1}^{\infty} \left[ \left( 1 - \frac{\alpha^{jk}}{1 - \alpha^k} \right) \alpha^j c_I + \alpha^{jk} c_P \right] (1 - \phi)^{(j-1)k} \left[ 1 - (1 - \phi)^k \right] - \\
\sum_{j=1}^{\infty} (c_I + c_P) \alpha^k \left[ \alpha (1 - \phi) \right]^{(j-1)k} \sum_{i=1}^{k} v_i(\phi, \theta) + \frac{c_F \sum_{i=1}^{k} \alpha^i v_i(\phi, \theta)}{1 - [\alpha (1 - \phi)]^k} \\
= \frac{\alpha^k c_I}{1 - \alpha^k} \left\{ 1 - \frac{\alpha^k \left[ 1 - (1 - \phi)^k \right]}{1 - [\alpha (1 - \phi)]^k} \right\} + \frac{\alpha^k c_P \left[ 1 - (1 - \phi)^k \right]}{1 - [\alpha (1 - \phi)]^k} -
\]
\[ \frac{\alpha^k (c_I + c_P) \sum_{i=1}^{k} v_i(\phi, \theta) \left( 1 - [\alpha (1 - \phi)]^k \right)}{1 - [\alpha (1 - \phi)]^k} + c_P \frac{\sum_{i=1}^{k} \alpha^i v_i(\phi, \theta)}{1 - [\alpha (1 - \phi)]^k} = \frac{\alpha^k \left( c_I + c_P \left[ 1 - (1 - \phi)^k \right] \right) + \sum_{i=1}^{k} \left[ \alpha^i c_P - \alpha^k (c_I + c_P) \right] v_i(\phi, \theta)}{1 - [\alpha (1 - \phi)]^k}. \]  

The expected “discounted cycle length”.

Similarly, the expected “discounted cycle length” becomes

\[ \sum_{i=1}^{\infty} \alpha^i p_i(\phi, \theta, k) = \sum_{j=1}^{\infty} \alpha^j \Pr\{\text{preventive repair at } jk \mid \phi, \theta\} + \sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \alpha^n \Pr\{\text{failure in } (n-1, n) \mid \phi, \theta\} \]

\[ = \sum_{j=1}^{\infty} \alpha^j (1 - \phi)^{(j-1)k} \Pr\{\text{preventive repair at } k \mid \phi, \theta\} + \sum_{j=1}^{\infty} \sum_{i=1}^{k} \alpha^{(j-1)k+i} (1 - \phi)^{(j-1)k} \Pr\{\text{failure in } (i-1, i) \mid \phi, \theta\} \]

\[ = \sum_{j=1}^{\infty} \alpha^j (1 - \phi)^{(j-1)k} \left\{ \left[ 1 - (1 - \phi)^k \right] - \sum_{i=1}^{k} v_i(\phi, \theta) \right\} + \sum_{j=1}^{\infty} \left[ \alpha (1 - \phi) \right]^{(j-1)k} \sum_{i=1}^{k} \alpha^i v_i(\phi, \theta) \]

\[ = \frac{\alpha^k \left[ 1 - (1 - \phi)^k \right] + \sum_{i=1}^{k} \left[ \alpha^i - \alpha^k \right] v_i(\phi, \theta)}{1 - [\alpha (1 - \phi)]^k}. \]  

The expected variable costs of a preventive repair per cycle \((c_V > 0)\).

The costs that remain to be determined are the expected variable costs of a preventive repair at time \(jk\Gamma\) where \(j, k \in \mathbb{N}\). Let \(j = 1\). If \(T_1 = i - 1 \Gamma 1 \leq i \leq k \Gamma \) then the variable repair costs are proportional to the amount of transported rock in the remaining time up to \(k \) not inducing a failure: i.e. \(X_{k-i+1} \leq y\). By taking the expectation of \(X_{k-i+1}\) (subject to \(X_{k-i+1} \leq y\)) \(\Gamma\) summing over all possible values for \(i\Gamma\) and using Theorem 1\(\Gamma\) we obtain

\[ E(\text{amount of transported rock in } (0, k] \text{ with preventive repair at } k \mid \phi, \theta) = \]
\[
E \left( \text{variable preventive repair costs per renewal cycle} \big| \phi, \theta \right) = \frac{\alpha^k c_V \left[ \xi_{k,k}(\phi) - \sum_{h=1}^{k+1} \xi_{k-h+2,k}(\phi) q_h(\theta) \right] \theta}{1 - \left[ \alpha (1 - \phi) \right]^k}
\]
References


