Non-Unique Flows in Macroscopic First-Order Intersection Models – An Equilibrium Theory

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Research on macroscopic node models is limited comparing to link models, but developed extensively in recent years.

- Node models been explored recently:
  - **Merge/diverge**: Daganzo, 1995; Jin and Zhang, 2003; and Ni and Leonard, 2005
  - **General applicability**: Lebacque and Khoshyaran, 2005; Tampère et al., 2011; Flötteröd and Rohde, 2011; Gibb, 2011; Corthout et al., 2012; Smits et al., 2015; Jabari, 2016

- **Non-uniqueness** is found to appear in some cases
  - Flötteröd and Rohde, 2011; Corthout et al., 2012
  - Undesirable feature for deterministic models
  - Insight needed into non-uniqueness and ways to deal with it
Background: general node model

- multiple **incoming links** and multiple **outgoing links** (E.g. signalized intersections)

- Input:
  - Demand - flow rate $\delta_i$ of incoming link $i$ + split rates $f_{io}$ to outgoing links $o$
  - Supply - available space $\sigma_o$ in outgoing link $o$

- Output:
  - Flow $q_{io}$ of each pair of incoming - outgoing link
Generic node model requirements

- holding-free solution: \((\delta_i - q_i) \prod_{o' \mid f_{io'} > 0} (\sigma_{o'} - q_{o'}) = 0\) \; \forall i

- feasible region:
  - non-negativity: \(q_{io} \geq 0\) \; \forall i
  - demand constraints: \(q_i = \sum_o q_{io} \leq \delta_i\) \; \forall i
  - supply constraints: \(q_o = \sum_i q_{io} \leq \sigma_o\) \; \forall o
  - CTF constraints: \(f_{io} = \frac{\delta_{io}}{\delta_i} = \frac{q_{io}}{q_i}\) \; \forall i, o
  - invariance principle: if \(\exists i \mid q_i < \delta_i\) \; \(q_i\) invariant for \(\delta_i \to C_i\)
  - (+ internal supply constraints)

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**Internal supply**

- **Internal supply**: additional capacity lost at conflict points: only one flow at a time
  - e.g. left turn yields to the oncoming traffic
  - left turn (2,6) not only determined by capacity of target link (6) but also by oncoming traffic (1,3)
  - represented as decreasing (convex) curve

- **Priority Ratio**: when the demand is high…
  - solution lies on Internal Supply Constraint
  - conventional solution: intersection with priority ratio line
    - (Originally in Daganzo (1995), “incremental transfer”)
  - for most intersections, performs well without any issue.
Non-Uniqueness

- Corthout et al. (2012) example with two left-turn movements yielding to oncoming traffic,
  - traditional method of priority ratio **fails** to find **unique solution**.
  - may also occur without internal supply constraints!

- Possible whenever different priority ratios apply to turns from same in-link:
  - Left-turn (2,6) and oncoming (1,3) → N
  - Left-turn (1,5) and oncoming (2,4) → M

- Multiple possible solutions:
  - M or N ??
Non-Uniqueness: problem summary and contribution

• different turning flows compete for limited supply (in outgoing links or internal conflict points) with **different priorities**
• unique solution for the turning flows cannot be guaranteed
• non-uniqueness not desired in deterministic dynamic network loading model
• to deal with this issue, **pre- or post-processing** approaches have been suggested by Corthout et al., (2012)
• we propose **equilibrium theory** which solves some non-unique cases without pre- or post-processing + identifies **all** solutions if not unique
• directions to solve remaining non-unique cases are identified (though not yet solved)
The Equilibrium Theory: another look at the priority ratio

- Intersect of **priority ratio** and a **constraint**
  - Intersect of \{ priority ratio and its corresponding constraint \} is behaviorally consistent, hence creates an **attractor**
  - Intersect of \{ priority ratio and other constraint(s) \} is **NO attractor**, rather activates constraint with (behavioral) tendency towards its corresponding priority ratio

- **Non-equilibrium flow ratios** induce behavior → attractor
  - minor flow suppressed too strongly
    → impatient minor flow drivers force themselves onto conflict (whether allowed by traffic regulations or not)
    → and/or polite major flow drivers let minor vehicles in
  - likewise with major flow suppressed too strongly

\[ q_{\text{minor}} \rightarrow q^{*} \rightarrow \hat{q} \]

\[ q_{\text{major}} \]

\( \bar{q} \): Major flows suppressed too strongly

\( \hat{q} \): Minor flows suppressed too strongly
The Equilibrium Theory

Same Solution as Conventional Point of View

- If solution is unique, **equilibrium point** \( q^* \) generates the same solution as the conventional point of view.
- Any arbitrary point \( \tilde{q} \) or \( \hat{q} \) on the (internal) supply constraint, follows chevrons, converging to the equilibrium (priority rate \( \alpha_{\text{minor}} / \alpha_{\text{major}} \)).
- The **theory** however also **solves** (some) former non-unique cases.

In this case:

- Same solution
- Conventional point of view
- Trying to restore equilibrium
If we would aim for $E_1$, 
- reaching M, flows are constrained stronger by constraint 2, which attracts towards $E_2$
- search direction is shown by chevrons. When reaching S, no other feasible point reachable by chevron(s) $\rightarrow$ solution S
- same reasoning applies to N
- regardless of start point, S is unique solution

Notes:
- M is just a point where ray 1 is intercepted because constraint 2 prevents it from reaching $E_1$
- M is NOT an attractor, so **in no way should it be the solution.**

This way of looking at M was not identified before
We solved some cases…

• previously thought to require pre- or post-processing
• Attractors lay outside feasible region
• Chevrons represent *dynamics* of users towards attractors
• aiming for attractors leads to the same point - \( S \), the solution.

... but some cases are not solved

• Multiple feasible attractors lie on the frontier \( \rightarrow \) multiple solutions/attractors
  • \( E_1 \) and \( E_2 \)
  • *chevrons* still represent the *dynamics*, but now indicate multiple directions.
Discussion: why sometimes unique and other cases not?

• Root cause of non-uniqueness: **cyclic priority**

• The role of **CTF** is: it acts as the **bridge** in constituting the cyclic nature.

\[ P_{(1,3)} \succ_p P_{(2,6)} \simeq_{CTF} P_{(2,4)} \succ_p P_{(1,5)} \simeq_{CTF} P_{(1,3)} \]

• **If one of CTF is zero**, or one of the priority directions would be inverse, this priority chain does not form a cycle, then there would be unique solution

• Likewise, **if one of CTF-bridges is too weak**, it may also allow unique solutions despite of the circularity
  - “too weak”? → remains to be further researched
Possible algorithms

- proposed equilibrium theory promising in devising algorithms, identifying:
  - unique solution if exists
  - all of multiple solutions

- Start from 0 and increase flow towards candidate attractor $E_i$
  - When hitting another constraint before reaching $E_i$, follow chevrons until
    - another constraint prevents this: this is a solution $S_i$
    - you end up in the corresponding $E$
  - otherwise, $E_i$ is a solution

- same for next $i$ until all $i$’s visited
  - when also ending in the already found $S_i$ or $E_i$, still unique
  - otherwise new non-unique solution added to the set
Possible algorithms

• If unique solution exist, then we’re done

• If multiple **feasible attractors**, then multiple solutions exist, two ways of post-processing:
  • In Microscopic way: For example, estimate the stationary value by Markov procedure in Flötteröd and Flötteröd (2013)
  • In Macroscopic way: calculate expected values
    \[
    \text{Expected state} = p[E_1] \cdot E_1 + p[E_2] \cdot E_2
    \]
    • in figure this is a linear combination

Expected Value of Multiple Solutions

Multiple solutions on convex constraints

• When non-unique solutions are identified
  • $q_1$-dominated constraint intersects its alpha-ratio ray showing attractor within the feasible region
  • $q_2$-dominated constraint intersects its alpha-ratio ray showing attractor within the feasible region

• Both **attractors feasible** -> **multiple solutions**
• The overall average effect might be in a intermediate position!

A linear combination between two feasible solutions may lie outside the feasible region!
Discussion on $p[E_1]$ and $p[E_2]$

- **$E_1$: $q_1$-dominated.**
  - On certain duration, we have solution at one (feasible) attractor ($E_1$)
  - But this may not last for too long:
    - Drivers from 2 would become impatient; Aggressive drivers emerge from 2; or Polite, considerate driver from 1
    - if sufficient of these behaviors occur, this shifts solution from $E_1$ to the other.

- **$E_2$: $q_2$-dominated:** similar reasoning
  - long duration (e.g. 1 hour), alternation between $E_1$ and $E_2$ with
    - expected value $p[E_1] \cdot E_1 + p[E_2] \cdot E_2$ of two feasible states may itself NEVER occur at any given time; may even lie outside feasible region
  - short duration (e.g. 1 min), in reality $E_1$ or $E_2$ would occur
    - still modeled as expected value?
    - Monte Carlo sample: stochastic model (cfr. microsimulation)?
Towards 3D-cases (→ N-dim?)

- Consider priority-to-the-right case (simplified to 3 approaches for sake of graphical representation)
  - root cause: *circular dependency*.
    - $P_{(1,5)} \succ_P P_{(2,6)} \succ_P P_{(3,4)} \succ_P P_{(1,5)}$
  - $→$ 3 sets of *priority ratios*, and they intersect with constraints at different points

- only intersects of constraints with their *corresponding* priority ratio are behavioral attractors
  - other intercepts are ordinary points on a constraint, deviating towards its priority ratio

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The 3D-case

- $q_2$ yields to $q_1 \rightarrow$ by combining *internal supply constraint* with its *priority ratio*, we get attractor $(q_1, q_2) = (a, b)$, then in 3D the attractor should be $(a, b, \infty)$ since this attractor should in no way bother $q_3$.

- Likewise, $q_3$ yields to $q_2$, and $q_1$ yields to $q_3$, with their attractors
A 3D-case with unique solution

- These constraints in 3D form a polyhedral
  - Here all attractors outside feasible region
  - Starting from red surface ($q_2$ yields to $q_1$), direction to attractor 1 points to $S$.
  - Intersections of surfaces yield lines with directions toward attractors also pointing toward $S$
  - Same reasoning applies starting in other two surfaces (green: $q_3$ yields to $q_2$. Orange: $q_1$ yields to $q_3$)
  - Regardless of starting point, final solution is $S \rightarrow$ unique
3D-case: uniqueness not guaranteed

- in certain combinations of (cyclic) priorities and (internal) supply constraints, multiple solutions may sustain, e.g.:
  - attractor favors $q_2$ instead of $q_1$
  - attractor favors $q_3$ instead of $q_2$
  - attractor favors $q_1$ instead of $q_3$

- Possibly undetermined search direction or cannot terminate
<table>
<thead>
<tr>
<th></th>
<th>No (0) attractor on feasible frontier</th>
<th>Single (1) attractor on feasible frontier</th>
<th>Multiple (2+) attractors on feasible frontier</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unique</strong> (U)</td>
<td><strong>U-0</strong>: Does not need pre- or post-processing to find <strong>unique solution</strong>. (so far considered non-unique in literature)</td>
<td>Unique solution.</td>
<td>Not possible because violates a necessary condition</td>
</tr>
<tr>
<td><strong>Non-unique</strong> (N)</td>
<td><strong>N-0</strong>: Calls for post-processing</td>
<td>(not observed, but still cannot proof inexistency)</td>
<td><strong>N-2+</strong>: Calls for post-processing</td>
</tr>
</tbody>
</table>

![Diagram](image1.png)

![Diagram](image2.png)
Discussion 3D-cases

• Case U-0:
  • reported with possibly non-unique solutions in literature
  • considering priority ratio as ratio of flow to restore equilibrium, intersecting points could be attractor or not.
  • In U-0, all points in the feasible region hit by the ray/plane of priority ratio are NO attractors
  • We now reach unique solution without additional treatment(s) (pre- nor post-processing)

• N-0, N-2+: Still requires post-processing
  We now understand the conditions in which this can happen + onset of algorithms to identify all solutions + suggestions of how to combine into unique solution (micro or macro)

Note: table does not include non-congested cases whose feasible region is bounded (trivially) by demand constraints.
• We re-interpreted previously proposed priority ratios at conflicts as attractors towards an empirical equilibrium behavior.

• This allowed us to find a unique solution in some pathological cases that formerly were suspected to have multiple solutions on a Pareto front.

• In other cases, a unique solution still cannot be guaranteed (if multiple attractors exist on the Pareto front). We can however identify when an intersection is prone to having multiple solutions and do the post treatment.

• Could easily see how behavior drifts the system either towards a unique solution, or towards multiple solutions.

• We identify some of the problem stems from cyclic priority. Once considered one of the root causes, CTF only acts as “bridge” in cyclic priority.
Closing (cont’d): future work

• General **N-dim cases** (e.g. 4-way yield-to-the-right crossings)

• Novel viewpoint on priority ratio → **devise efficient algorithms**
  • identifying all possible solutions (unique or not)

• solid and theoretically justified **post-processing** of multiple solutions

• **Empirical validation**: do observations confirm our theories?
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• Any questions/comments?

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