Aide Mémoire

Subject: Useful formulas for sediment transport and morphology in rivers and channels

The University cannot take responsibility for any misprints or errors in the presented formulas. Please use them carefully and wisely.

1. General basic equations

Three-dimensional (3D):

The general 3D mass-balance equations for a unit volume of 2-phase (fluid and sediment) flow (after time-averaging of turbulence and applying the eddy-viscosity concept) are:

For fluid:

\[
\frac{\partial (1 - c_s)}{\partial t} + \frac{\partial u (1 - c_s)}{\partial x} + \frac{\partial v (1 - c_s)}{\partial y} + \frac{\partial w (1 - c_s)}{\partial z} + \nabla \left( \frac{\epsilon_{fs}}{\mu} \frac{\partial c_s}{\partial x} \right) + \nabla \left( \frac{\epsilon_{fs}}{\mu} \frac{\partial c_s}{\partial y} \right) + \nabla \left( \frac{\epsilon_{fs}}{\mu} \frac{\partial c_s}{\partial z} \right) = 0
\]

For sediment (convection-diffusion equation):

\[
\frac{\partial c_s}{\partial t} + \frac{\partial uc_s}{\partial x} + \frac{\partial vc_s}{\partial y} + \frac{\partial w(c_s - w_s)}{\partial z} + \nabla \left( \frac{\epsilon_{ss}}{\mu} \frac{\partial c_s}{\partial x} \right) - \nabla \left( \frac{\epsilon_{ss}}{\mu} \frac{\partial c_s}{\partial y} \right) - \nabla \left( \frac{\epsilon_{ss}}{\mu} \frac{\partial c_s}{\partial z} \right) = 0
\]

where:

- \(c_s\) = local mean sediment volume concentration (suspension)
- \(u, v, w\) = fluid-velocity in \(x, y, z\) -direction
- \(w_s\) = fall velocity of sediment particles
- \(\epsilon_{fs}, \epsilon_{ss}, \epsilon_{st}\) = fluid mixing coefficient in \(x, y, z\) -direction
- \(\epsilon_{sx}, \epsilon_{sy}, \epsilon_{sz}\) = sediment mixing coefficient in \(x, y, z\) -direction (assume \(\epsilon_{s} = \epsilon_{f}\))

Or after rewriting:
\[
\frac{\partial c_i}{\partial t} + u \frac{\partial c_i}{\partial x} + v \frac{\partial c_i}{\partial y} + w \frac{\partial c_i}{\partial z} - \frac{\partial}{\partial y} \left[ \epsilon_{xy} \frac{\partial c_i}{\partial y} \right] - \frac{\partial}{\partial z} \left[ \epsilon_{xz} \frac{\partial c_i}{\partial z} \right] = 0
\]

(3)

where

\[ \alpha = \text{coefficient used for large concentrations (small concentrations: } \alpha = 1, \text{ else } \alpha \approx 4 \text{ to } 5) \]

Two-dimensional depth-averaged (2DH)

The general 2D continuity equations for sediment, after depth-averaging, become:

Sediment balance:

\[
\left(1 - \epsilon_p\right) \frac{\partial z}{\partial t} + \frac{\partial s_{bx}}{\partial x} + \frac{\partial s_{by}}{\partial y} + \frac{\partial hC_i}{\partial t} + \frac{\partial U hC_i}{\partial x} + \frac{\partial V hC_i}{\partial y} = 0
\]

(4)

where:

- \( C_s \) = depth-averaged suspended sediment concentration
- \( h \) = water depth
- \( s_{bx}, s_{by} \) = bed-load transport per unit of width in \( x,y \)-direction
- \( U, V \) = depth-averaged flow velocity in \( x,y \)-direction
- \( \epsilon_p \) = porosity of bed (void volume / total volume) (for sand 0.4 ± 0.05)

Modified Galappatti’s equation for suspended sediment:

\[
T' a \frac{\partial C_i}{\partial t} + L' a U \frac{\partial c_i}{\partial x} + L' a V \frac{\partial c_i}{\partial y} = \frac{w_s}{h} (C_{se} - C_s)
\]

(5)

where:

- \( T' a \) = dimensionless adaptation time scale (roughly between 0.5 and 1)
- \( L' a \) = dimensionless adaptation time scale (approximately equal to \( T' a \))
- \( C_s \) = depth-averaged equilibrium bed-load concentration
- \( C_{se} \) = depth-averaged equilibrium suspended sediment concentration (following from transport equation)

The adaptation time scale can be obtained from the figure below, as function of the ratios \( w_s/u \) and \( u/\bar{u}=(\sqrt{g}/C). \) Dependent on the type of boundary condition used at the bed for depth-integration a value for \( T' a \) can be selected from the graphs below. The concentration type is more suitable during erosion, while the other works better during sedimentation.
Figure 1 Galappatti’s adaptation time-scale for concentration-type boundary condition (left) and gradient-type boundary conditions (right)

One-dimensional cross-sectional averaged (1D)

The one-dimensional basic equation for sediment can be obtained directly from the 2DH equations, resulting in:

\[
\frac{1 - \varepsilon_p}{\varepsilon_p} \frac{\partial z_h}{\partial t} + \frac{\partial s_{bh}}{\partial x} + \frac{\partial h C_z}{\partial t} + \frac{\partial U h C_z}{\partial x} = 0
\]  

(6)

And with the original Galappatti’s equation for suspended sediment:

\[
\frac{\partial C_z}{\partial t} + U \frac{\partial C_z}{\partial x} = \frac{w_s}{T'_a h} (C_{w} - C_z)
\]  

(7)

Time scale \( T'_a \) can be obtained from figure 1.

2. Sediment characteristics

Definitions and standards:

- Apply the Phi-size for plotting the sediment size distribution (based on assumption of log-normal distributions):

\[
\phi_i = \log_2 (D_i) \quad \text{with} \quad D_i \text{ in millimeter}
\]  

(8)

- Mode = most common grain size in the population (with two humps: bimodal mixture)

- Median grain size = \( D_{50} \) = grain size for which 50% of the sample is finer.

- Mean grain size = \( D_m \) = average grain size of the deposit with \( D_m = \sum p_i D_i \)

- Sieve diameter: if a particle falls through sieve \( D_a \) and remains on next sieve \( D_b \) then the sieve diameter \( D = (D_bD_a)^{1/2} \). Sieving is used for gravel and sand > 70 µm.

- Geometric mean size (first moment of size frequency distribution):
1. \[ D_g = \sqrt{D_{84}D_{16}} \] 

2. \[ D_g = 2^{-\phi_m} \quad \text{with} \quad \phi_m = \sum_{i=1}^{N} \phi_m p_i \] 

where \(\phi_m\) is the arithmetic mean phi-size, \(\phi_{mi}\) is the class middle (in between sieve mesh sizes) of a size fraction, and \(p_i\) is probability in size fraction.

- Geometric standard deviation (second moment of size frequency distribution; how large is variation around the mean):

1. \[ \sigma_g = 0.5 \left( \frac{D_{84}}{D_{50}} + \frac{D_{50}}{D_{16}} \right) \] 

2. \[ \sigma_g = 2^{\sigma_m} \quad \text{with} \quad \sigma_m = \sum_{i=1}^{N} (\phi_{mi} - \phi_m)^2 p_i \] 

where \(\phi_m\) is the arithmetic mean phi-size (see above), \(\phi_{mi}\) is the class middle (in between sieve mesh sizes) of a size fraction, and \(p_i\) is probability in size fraction.

- Specific weight of a deposit is the density or weight per unit volume, expressed as dry weight or dry density including the pores:

\[ \rho_d = \left(1 - \varepsilon_p\right) \rho_s \] 

with

\[ \varepsilon_p = \text{porosity (void volume / total volume) (for sand 0.4 \pm 0.05)}; \]

\[ \rho_s = \text{specific weight or density of sediment particles (usually order of 2650 kg/m}^3, \text{quartz)} \]

- Sediment concentrations are often expressed in milligrams per liter (\(C_{mg/l}\) in mg/l) or in parts per million (\(C_{ppm}\) in ppm). Conversion formulas:

- If \(C_{mg/l} < 16,000 \text{ mg/l}\) then: \(C_{mg/l} = C_{ppm}\)  \hspace{3cm} (12)

- If \(C_{mg/l} > 16,000 \text{ mg/l}\) then:

\[ C_{ppm} = \frac{10^6}{C_{mg/l} + 1 - \frac{1}{\sigma_s}} \quad \text{and} \quad C_{mg/l} = \frac{10^6}{C_{ppm} - 1 + \frac{1}{\sigma_s}} \] 

(13)

In which \(\sigma_s\) is the specific gravity of sediment (in the order of 2.65).
- Grain size classification

<table>
<thead>
<tr>
<th>Sediment</th>
<th>Millimetres</th>
<th>Sediment</th>
<th>Millimetres</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very large boulders</td>
<td>4096 - 2048</td>
<td>Very coarse sand</td>
<td>2.0 - 1.0</td>
</tr>
<tr>
<td>Large boulders</td>
<td>2048 - 1024</td>
<td>Coarse sand</td>
<td>1.0 - 0.5</td>
</tr>
<tr>
<td>Medium boulders</td>
<td>1024 - 512</td>
<td>Medium sand</td>
<td>0.5 - 0.25</td>
</tr>
<tr>
<td>Small boulders</td>
<td>512 - 256</td>
<td>Fine sand</td>
<td>0.25 - 0.125</td>
</tr>
<tr>
<td>Large cobbles</td>
<td>256 - 128</td>
<td>Very fine sand</td>
<td>0.125 - 0.062</td>
</tr>
<tr>
<td>Small cobbles</td>
<td>128 - 64</td>
<td>Coarse silt</td>
<td>0.062 - 0.031</td>
</tr>
<tr>
<td>Very coarse gravel</td>
<td>64 - 32</td>
<td>Medium silt</td>
<td>0.031 - 0.016</td>
</tr>
<tr>
<td>Coarse gravel</td>
<td>32 - 16</td>
<td>Fine silt</td>
<td>0.016 - 0.008</td>
</tr>
<tr>
<td>Medium gravel</td>
<td>16 - 8</td>
<td>Very fine silt</td>
<td>0.008 - 0.004</td>
</tr>
<tr>
<td>Fine gravel</td>
<td>8 - 4</td>
<td>Coarse clay</td>
<td>0.004 - 0.002</td>
</tr>
<tr>
<td>Very fine gravel</td>
<td>4 - 2</td>
<td>Medium clay</td>
<td>0.002 - 0.001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fine clay</td>
<td>0.0010 - 0.0005</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Very fine clay</td>
<td>0.0005 - 0.00024</td>
</tr>
</tbody>
</table>

- Fall velocity \( w_s \) [m/s] for sediment (van Rijn, 1984)

1. For Stokes range (\( Re=w_s D/\nu < 1 \) or \( D \leq 100 \) \( \mu \)m): \( w_s = \frac{1}{18} \frac{\Delta g D^2}{\nu} \) (14)

2. For 100 < \( D \leq 1000 \) \( \mu \)m: \( w_s = \frac{10\nu}{D} \left( \sqrt{1 + \frac{0.01 \Delta g D^3}{\nu^2}} - 1 \right) \) (15)

3. For \( D > 1000 \) \( \mu \)m: \( w_s = 1.1 \sqrt{\Delta g D} \) (16)

With
\( \Delta = \) relative density of sediment, defined as \( \frac{\rho_s - \rho_w}{\rho_w} \) (usually about 1.65)
\( \nu = \) kinematic viscosity for clear fluid (usually \( 1 \cdot 10^{-6} \) m\(^2\)/s)

3. Sediment transport

Classification

- To study sediment transport processes it is possible to classify the transport processes according to origin and mechanisms as in the following scheme (Jansen et al., 1979):
Shields and initiation of motion (Shields curve)

- The Shields parameter is defined as the ratio between flow forces and gravity forces on a sediment particle, and can be expressed as:

\[
\theta = \frac{\tau_b}{(\rho_s - \rho) g D} = \frac{u_*^2}{\Delta g D} = \frac{u_*^2}{C^2 \Delta} = \frac{h \cdot i}{\Delta D}
\]  

(17a)

where

- \( D \) = characteristic grain size
- \( i \) = slope
- \( u_* \) = shear velocity = \( (\tau_b / \rho)^{\frac{1}{2}} = u \cdot g^{\frac{1}{2}} / C \)
- \( \Delta \) = relative density = \((\rho_s - \rho) / \rho \approx 1.65 \) for quartz
- \( \rho \) = density of water \( \approx 1000 \text{ kg/m}^3 \)
- \( \rho_s \) = density of sediment \( \approx 1650 \text{ kg/m}^3 \) for quartz

- Initiation of motion according to Shields: the threshold for initiation of motion is defined by the critical Shields parameter in the Shields curve. The Shields curve expresses the relation between the Shields parameter \( \theta_c \) and the grain Reynolds number \( Re_c \), see figure:

\[
Re_c = \frac{u_* D}{\nu} ; \quad \theta_c = \frac{\tau_{cr}}{(\rho_s - \rho) g D} = \frac{u_{cr}^2}{\Delta g D}
\]  

(17b)

where

- \( Re_c \) = particle Reynolds number
- \( u_{cr} \) = critical shear velocity = \( (\tau_{cr} / \rho)^{\frac{1}{2}} \)
- \( \Delta \) = relative density = \((\rho_s - \rho) / \rho \approx 1.65 \) for quartz
- \( \nu \) = kinematic viscosity, usually \( 1 \cdot 10^{-6} \text{ m}^2/\text{s} \)
- \( \rho \) = density of water \( \approx 1000 \text{ kg/m}^3 \)
- \( \rho_s \) = density of sediment \( \approx 1650 \text{ kg/m}^3 \) for quartz
Van Rijn (1984) gives an approximation of the Shields curve as follows:

\[
\begin{align*}
\theta_{cr} &= \frac{0.24}{D_\tau} \quad \text{if } D_\tau \leq 4 \\
\theta_{cr} &= \frac{0.14}{D_\tau^{0.64}} \quad \text{if } 4 < D_\tau \leq 10 \\
\theta_{cr} &= \frac{0.04}{D_\tau^{0.10}} \quad \text{if } 10 < D_\tau \leq 20 \\
\theta_{cr} &= 0.013 D_\tau^{0.29} \quad \text{if } 20 < D_\tau \leq 150 \\
\theta_{cr} &= 0.055 \quad \text{if } D_\tau > 150
\end{align*}
\]

\[
\theta_{cr} = \frac{\mu u_{*} \Delta D}{\nu} \quad \text{with } D_\tau = D_{50} \left( \frac{\Delta g}{\nu^2} \right)^{1/3}
\] (18)

**Transport formula of Meyer-Peter and Müller (1984)**

The empirical formula of Meyer-Peter and Müller (MP-M) for bed-load transport is written as

\[
\Phi_s = 8 \left( \psi - 0.047 \right)^{3/2}
\] (19)

with

\[
\mu = \left( \frac{C}{C_{90}} \right)^{3/2} \quad \text{(ripple factor)}
\] (20)

\[
\Phi_s = \frac{s}{D^{3/2} \sqrt{g \Delta}} \quad \text{(transport parameter)}
\] (21)

\[
\psi = \mu \theta = \frac{\mu h_i \Delta D_{50}}{\Delta D_{50}} = \mu \frac{u^2}{C^2} \Delta D_{50} \quad \text{(flow parameter)}
\] (22)

\[
D = \overline{D} = \sum_i \left( p_i D_i \right) / \sum_i p_i
\] (23)
\[ C_{w0} = 18 \cdot \log \left( 12 \frac{h}{D_{w0}} \right) \quad (24) \]

\[ i = \text{energy gradient (bed gradient in case of uniform flow conditions)} \]

\[ s = \text{bed-load transport per unit of width (without pores, otherwise multiply with } \frac{1}{(1-\varepsilon_p) = 1.66}) \]

It is valid for situations in which \( w_s/u > 1 \), \( D_m > 0.4 \) mm, and \( \mu \theta < 0.2 \).

For the MP-M formula an approximate power function \( s = m u^n \) can be defined, with \( n \) equal to

\[ n = \frac{3}{1 - 0.047 \Psi^3} \quad (25) \]

Note: can only be used if \( C \)-value is known.

**Transport formula of Engelund and Hansen (1967)**

The formula of Engelund and Hansen (1967) for total load (bed- and suspended bed-material load) is written as:

\[ \phi_s = 0.05 \Psi^{5/2} \quad (26) \]

with

\[ \mu = \left( \frac{C^2}{g} \right)^{2/5} \text{ (ripple factor)} \quad (27) \]

\[ \Phi_s = \frac{s}{D_{s0}^{3/2} \sqrt{g \Delta}} \text{ (transport parameter)} \quad (\text{cf. eq. 21}) \]

\[ \Psi = \mu \theta = \frac{\mu hi}{\Delta D_{s0}} \text{ (flow parameter)} \quad (\text{cf. eq. 22}) \]

\[ i = \text{energy gradient (bed gradient in case of uniform flow conditions)} \]

\[ s = \text{bed-load transport per unit of width (without pores, otherwise multiply with } \frac{1}{(1-\varepsilon_p) = 1.66}) \]

The formula can also be written as (useful for unsteady flow):

\[ s = 0.05 \sqrt{g \Delta D_{s0}^3 \left( \frac{u_s}{\sqrt{g \Delta D_{s0}}} \right)^3 \left( \frac{u}{\sqrt{g \Delta D_{s0}}} \right)^2} = 0.05 \frac{u^5}{\sqrt{g C^3 \Delta^3 D_{s0}}} \quad (28) \]

Engelund and Hansen is valid for situations in which \( w_s/u < 1 \), \( 0.19 < D_{s0} < 0.93 \) mm, and \( 0.07 < \theta < 6 \).
Transport formula of van Rijn (1984)

The transport formula of van Rijn (1984) distinguishes between a bed-load part \( s_b \) and a suspended-load part \( s_s \):

\[
s = s_b + s_s
\]

(29)

A transport stage parameter \( T \) is defined as:

\[
T = \frac{\tau'_b - \tau_{b \text{cr}}}{\tau_{b \text{cr}}} = \frac{(C/C')^2 \theta - \theta_{cr}}{\theta_{cr}}
\]

(30)

with

- \( \tau'_b \) = bed-shear related to grains
- \( \tau_{b \text{cr}} \) = critical bed-shear according to the Shields curve

Bed-shear \( \tau'_b \) is written, with \( C=C_90 \) (see Meyer-Peter and Müller) as

\[
\tau'_b = \left( \frac{C}{C'} \right)^2 \tau_b
\]

(31)

A dimensionless particle parameter \( D \) is defined as

\[
D = D_50 \left( \frac{\Delta g}{\nu} \right)^{-\frac{1}{3}}
\]

(32)

Bed-load transport then follows from

\[
\Phi_b = 0.053 \frac{T^{2.1}}{D_s^{0.3}} \text{ voor } T < 3
\]

\[
\Phi_b = 0.1 \frac{T^{1.5}}{D_s^{0.3}} \text{ voor } T \geq 3
\]

(33)

with a transport parameter

\[
\Phi_b = \frac{s_b}{\sqrt{g \Delta D_{50}^3}}
\]

(34)

and \( s_b \) is bed-load transport without pores, particles in the range 200-2000 \( \mu m \).

Suspended-load transport then follows from

\[
s_s = Fu h c_u
\]

(35)

with

- \( u \) = depth-averaged flow velocity
- \( h \) = water depth
$F = \text{integration factor}$

$C_a = \text{reference concentration at level } a \text{ measured from the bed}$

$a = \text{reference level, assumed to be } 1/2*\text{bed-form-height or equal to equivalent roughness height } k$, with minimum value $0.01*h$. 

The reference concentration $c_a$ (excluding pores) is written as

$$c_a = 0.015 \frac{D_{50}}{a} T^{1.5}$$

(36)

And the integration factor $F$ is written as:

$$F = \frac{\left( \frac{a}{h} \right)^{Z'} - \left( \frac{a}{h} \right)^{1.2}}{\left( 1 - \frac{a}{h} \right) (1.2 - Z')}$$

(37)

in which

$$Z' = \frac{w_s}{1 + 2 \left( \frac{w_s}{u_*} \right)^2} + 2.5 \left( \frac{w_s}{u_*} \right)^{0.8} \left( \frac{c_a}{0.65} \right)^{0.4} \text{ for } 0.01 \leq \frac{w_s}{u_*} \leq 1$$

(38)

Where $\kappa = \text{constant of Von Karman } (=0.4)$. The fall velocity $w_s$ is computed (using equations 14 to 16) from the representative grain size of suspended sediment $D_s$, following from

$$\frac{D_{s}}{D_{50}} = 1 + 0.011 \left( \frac{1}{2} \left( \frac{D_{s}}{D_{50}} + \frac{D_{50}}{D_{16}} \right) - 1 \right)(T - 25)$$

(39)

**Transport formula of Ackers and White (1973)**

The formula of Ackers and White (1973) starts from the definition of $D_{gr}$:

$$D_{gr} = D_{50} \left[ \frac{g A}{V^2} \right]^{\frac{1}{3}}$$

(40)

for coarse sediment, which is only considered here if $D_{gr} > 60$.

Then can be deduced that

$$G_{gr} = 0.025 \left[ \frac{F_{gr}}{0.17} - 1 \right]^{1.5}$$

(41)

with
\[ F_{gr} = \frac{1}{\sqrt{g\Delta D_{35}}} \frac{u}{\sqrt{32 \log \frac{10h}{D_{35}}}} \]  \hspace{1cm} (42)

and

\[ X_s = \frac{(\Delta + 1)D_{35}}{h} G_{35} \left\{ \frac{C}{\sqrt{g}} \right\} \]  \hspace{1cm} (43)

and finally

\[ s = \frac{X_q \rho}{\rho_s (1-\varepsilon)} \]  \hspace{1cm} \text{met } \varepsilon \approx 0.4 \hspace{1cm} (44)
4. Bed forms and alluvial roughness

Types and classification are shown below:

<table>
<thead>
<tr>
<th>Flow regime</th>
<th>Bedform</th>
<th>Bed-material concentration (ppm)</th>
<th>Mode of sediment transport</th>
<th>Type of roughness</th>
<th>Roughness $\frac{C}{\sqrt{g}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower regime</td>
<td>Ripples</td>
<td>10 – 200</td>
<td>Discrete steps</td>
<td>Form roughness predominate</td>
<td>7.8 – 12.4</td>
</tr>
<tr>
<td></td>
<td>Ripples on dunes</td>
<td>100 – 1,200</td>
<td></td>
<td></td>
<td>7.0 – 13.2</td>
</tr>
<tr>
<td></td>
<td>Dunes</td>
<td>200 – 2,000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Transition</td>
<td>Washed-out dunes</td>
<td>1,000 – 3,000</td>
<td>Variable</td>
<td></td>
<td>7.0 – 20.2</td>
</tr>
<tr>
<td>Upper regime</td>
<td>Plane bed</td>
<td>2,000 – 6,000</td>
<td>continu</td>
<td>Grain-roughness predominates</td>
<td>16.3 – 20</td>
</tr>
<tr>
<td></td>
<td>Antidunes</td>
<td>&gt; 2,000</td>
<td></td>
<td></td>
<td>10.3 – 20</td>
</tr>
<tr>
<td></td>
<td>Chutes and pools</td>
<td>&gt; 2,000</td>
<td></td>
<td></td>
<td>9.4 – 10.7</td>
</tr>
</tbody>
</table>

**Alluvial roughness predictor of White c.s (1980)**

The roughness predictor of White C.S. (1980), here only presented for coarse material $(D_{90} = D_{35}[g\Delta(V^2)]^{1/3} > 60)$, follows an iteration procedure. Given is bed slope $i$, a discharge per unit of width $q$, grain characteristics ($\rho_s$ and $D_{35}$), and fluid characteristics ($\rho$ and $\nu$). The water depth ($h$) is estimated, from which can be computed that $u_* = \sqrt{ghi}$.

This results in
\[ F_{fs} = \frac{u}{\sqrt{g\Delta D}} \]  

(45)

Then determine \( F_{gr} \) from:

\[ \frac{F_{gr} \cdot 0.17}{F_{fs} \cdot 0.17} = 1.0 - 0.76 \left[ 1.0 - \exp \left( \log D_{gr} \right)^{1.7} \right] \]  

(46)

Compute the flow velocity \( u \) from

\[ F_{gr} = \frac{1}{\sqrt{g\Delta D}} \frac{u}{\sqrt{32 \cdot \log\{10h / D\}}} \]  

(47)

Make an improved estimate for the depth \( h = q / u \) until \( h \) is sufficiently accurate.

Compute \( C \) from

\[ C = \frac{u}{u} = \frac{u}{\sqrt{h}} \]  

(48)

τ) in a component due to bed forms (\( \tau' \)) and a component due to roughness of sediment particles (\( \tau \)). Hence,

\[ hi = (hi)'' + (hi)' \]  

(49)

with \( i = \tau' = i' \), the depth \( (h) \) is subdivided into \( h' \) and \( h'' \).

In 1967 Engelund en Hansen propose (with \( \psi = h i / \Delta D \) en \( \varphi = hi / \Delta D \)):

\[ \Psi' = 0.06 + 0.4\psi^2 \]  

(50)

However, based on measurements, Engelund and Fredsøe (1982) propose:

\[ \Psi' = 0.06 + 0.3\psi^{3/2} \]  

(51)

With given \( q, i, \Delta \) and \( D \) the iteration procedures is as follows:

(i) Estimate \( u \), after which \( h' \) can be computed from

\[ \frac{u}{u_c} = \frac{u}{\sqrt{g hi}} = 9.45 \left\{ \frac{h'}{k} \right\}^{1/8} \]  

(52)

i.e., for roughness without bed forms.

Recommended value for \( k = 2.5\cdot D_s \) (sedimentation diameter).

(ii) Determine \( \psi = \psi' / \Delta D \)

(iii) Determine \( \varphi \) from eq. (50) or (51) respectively. From \( \varphi \) follows \( h \).
(iv) Make an improvement estimation of $u$ from $q = u \cdot h$ and repeat the procedure.

(v) With $u = C\sqrt{h}$ the resulting $C$-value can then be determined.

**Roughness predictor of Van Rijn (1987)**

The bed form height $H$ can be estimated from the following equation:

$$H = 0.11 h \left( \frac{D_{50}}{h} \right)^{0.3} \left( 1 - e^{-0.5T} \right) (25 - T) \quad \text{if} \quad 0 < T < 25$$

$$H = 0 \quad \text{if} \quad T \leq 0 \quad \text{or} \quad T \geq 25$$

In which:

$T = \text{transport stage parameter, defined by equation (30)}$

$h = \text{water depth}$

Furthermore, the bed form length $L$ can be defined as:

$$L = 7.3 h$$

(54)

Based on these formules the total equivalent Nikuradse roughness height $k_s$ is defined by van Rijn as:

$$k_s = k'_s + 1.1 H \left( 1 - e^{-2sH} \right)$$

(55)

In which

$k'_s = 3 \ D_{50}$ is the equivalent Nikuradse roughness height of grains.

Using the equivalent Nikuradse roughness height the Chézy coefficient for total roughness becomes

$$C = 18 \log \left( \frac{12 h}{k_s} \right)$$

(56)
5. **Morphological time scales and celerities**

The celerities express the propagation speed of small disturbances (infinitely small waves) on the water surface or on the bed. For the one-dimensional equations three celerities can be deduced:

- a negative and a positive celerity \((c_{1,2})\) associated to the propagation of long waves on the water surface \((\approx u \pm \sqrt{gh})\)

- a positive celerity associated to the propagation of small bed-waves, with an approximate value:

  \[
  c_s = \left( \frac{u_m \psi}{1 - Fr^2} \right) \text{ where } \psi = \frac{ds}{du} = n \cdot \frac{s}{q}
  \]  

  \((57)\)

  with \(Fr\) = Froude number of the main-channel flow, \(h\) = average water depth in main channel, \(q\) = main-channel discharge per unit of width, \(s\) = sediment-transport rate per unit of width, \(u_m\) = flow velocity in main channel, \(n\) = power of the transport formula \((If \ s = mu^n)\).

Theoretically the morphological time-scale \(T\) is defined as

\[
T = \varepsilon \frac{L \cdot h}{s}
\]  

\((58)\)

With \(\varepsilon\) = definition factor dependent on the actual time-scale definition, with an order of magnitude of 1.0; and \(L\) = characteristic length scale.

Different types of phenomena are usually considered for definition of the time scale:

- Wave propagation: for small length scales hydraulic roughness is negligible, and morphological changes have a wave-character. The propagation speed of the wave (with small amplitude) is defined as:

  \[
  c = \frac{u}{h} \left( \frac{ds}{du} \right) = \frac{n \cdot s}{h \left( 1 - Fr^2 \right)}
  \]  

  \((59)\)

  With \(n\) = non-linearity of the transport formula \((e.g., \text{see formula 25})\), and \(Fr\) is the Froude number. The time scale \(T_g\) is defined as the time in which the wave propagates of a distance \(L_g\):

  \[
  T_g = \frac{L_g}{c} = \frac{L_g h \left( 1 - Fr^2 \right)}{n \cdot s}
  \]  

  \((60)\)

- Diffusion: for changes over long lengths \((L_d > \{2 \text{ to } 3\} \cdot h/i)\) the bed-profile development has a diffusion character with a diffusion coefficient

  \[
  D = \frac{u}{3i} \left( \frac{ds}{du} \right) = \frac{n \cdot s}{3i}
  \]  

  \((61)\)

  With \(i\) = slope. The time scale \(T_d\) is defined as the time in which, at a distance \(L_d\) upstream of a measure, 50% of the final erosion or sedimentation due to this measure has occurred

  \[
  T_d = \frac{L_d^2}{D} = \frac{3L_d^2 \cdot i}{n \cdot s}
  \]  

  \((62)\)
Relaxation of 2D (depth-averaged) processes: for processes that adapt gradually to an equilibrium condition, resulting in a diffusion-type of adaptation of the 2D morphology. The time-scale $T_{2D}$ is defined as the time in which along one bank 50% of the total erosion or sedimentation occurs, caused by a measure taken on the other bend. It is written as:

$$T_{2D} = \frac{B^2}{\pi^2 s} \frac{f(\theta)}{s}$$  \hspace{1cm} (63)

Where \( B \) = river width, \( f(\theta) \) = function of the Shields parameter which expresses the effect of transverse bed-slope on the direction of sediment transport (see section 6), and \( \lambda_s \) = adaptation length for 2D morphology according to Struiksma et al (1985), defined as:

$$\lambda_s = \frac{f(\theta)}{\pi^2} \left( \frac{B}{h} \right) B$$  \hspace{1cm} (64)

### 6. Two-dimensional (2D) morphology

Due to curvature of flow (in river bends) the morphology is determined by near-bed flow deflection by helical flow and by gravity effects on particles by transverse slope. These effects are reproduced by correcting the sediment-transport direction as follows:

$$\tan(\beta) = \frac{\sin(\alpha) - f(\theta, \eta) \frac{\partial z}{\partial \eta}}{\sin(\alpha) - f(\theta, \xi) \frac{\partial z}{\partial \xi}}$$  \hspace{1cm} (65)

In which \( \partial z/\partial \eta \) and \( \partial z/\partial \xi \) are bed-level slope in transverse \( (\eta) \) and in flow \( (\xi) \) direction respectively, and \( f(\theta) \) = function of the Shields parameter which expresses the effect of transverse bed-slope on the direction of sediment transport.

In river bend redistribution of flow and sediment occur, which leads to an axi-symmetric solution in a long bend (or a bend with strongly damped morphology) with constant radius. The transverse bed slope \( \beta \) of the axi-symmetric situation is defined as:

$$\tan(\beta) = A f(\theta) \frac{h}{R}$$  \hspace{1cm} (66)

In which \( R \) = radius of curvature of the bend flow, and \( A \) = spiral flow coefficient defined as:

$$A = \frac{2\varepsilon}{k^2} \left( 1 - \frac{\sqrt{g}}{kC} \right)$$  \hspace{1cm} (67)

Where \( \varepsilon \) = calibration factor \((-1)\), \( k \) = Von Karman coefficient \((=0.4)\), \( g \) = gravity acceleration, and \( C \) = Chézy value. The function \( f(\theta) \) was defined by Talmon et al (1995) as:

$$f(\theta) = 9 \left( \frac{D}{h} \right)^{0.3} \sqrt{\theta}$$  \hspace{1cm} (68)

In practice often the function is written as \( f(\theta)=0.85\sqrt{\theta} \) for natural channels (or about \( 1.7\sqrt{\theta} \) for laboratory flumes).

The shape of a point bar in river bend is determined by the wave length \( L_p \) and damping length \( L_D \) expressed by Struiksma et al (1985) as:
\begin{align*}
2\pi \frac{\lambda_w}{L_p} &= \frac{1}{2} \sqrt{(n+1)IP^{-1} - IP^{-2} - \left(\frac{n-3}{2}\right)^2} \\
\frac{\lambda_w}{L_D} &= \frac{1}{2} \left(IP^{-1} - \frac{n-3}{2}\right)
\end{align*}

(68) (69)

Where \( n \) = non-linearity of the transport formula (e.g., see formula 25), \( \lambda_w \) = adaptation length for 2D flow, defined as \( \lambda_w = C^2 h (2g) \), \( \lambda_s \) = adaptation length for 2D morphology according to Struiksma et al (1985), defined as equation 64. IP is the interaction parameter defined as:

\[
IP = \frac{\lambda_s}{\lambda_w} \approx \frac{g}{C^2} \left(\frac{B}{h}\right)^2 \left(\frac{D}{h}\right)^{0.3} \sqrt{\theta}
\]

(70)