Max-plus-linear systems

Ton van den Boom

Introduction week 2018
Algebra

Conventional \((+, \times)\)-algebra:

\[(a + b) \times c = a \times c + b \times c\]
\[a \times (b + c) = a \times b + a \times c\]
\[(\lambda \times a) \times (\mu \times b) = (\lambda \times \mu) \times (a \times b)\]

Introduce a different algebra:

\[+ \quad \Longrightarrow \quad \text{max}\]
\[\times \quad \Longrightarrow \quad +\]
Max-Plus Algebra

Introduce notation from max-plus algebra:

\[ x \oplus y = \max(x, y) \quad x \otimes y = x + y \]

Matrices:

\[
\begin{align*}
[A \oplus B]_{ij} &= a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}) \\
[A \otimes C]_{ij} &= \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1, \ldots, n} (a_{ik} + c_{kj})
\end{align*}
\]
Max-plus-linear discrete-event systems

System description:

\[
\begin{align*}
x(k) &= A \otimes x(k - 1) \oplus B \otimes u(k) \\
y(k) &= C \otimes x(k)
\end{align*}
\]

\(k\) is an event counter
\(x_i(k)\) is time that state event \(i\) occurs in \(k\)th cycle.
\(u(k)\) is time that input event occurs in \(k\)th cycle.
\(y(k)\) is time that output event occurs in \(k\)th cycle.
Modeling of railway networks

Railway networks can be modeled using max-plus linear models.
$x_1 =$ departure time of train 1 from station A

$x_2 =$ departure time of train 2 from station D

$x_3 =$ departure time of train 1 from station B

$d_3 =$ scheduled departure time of train 1 from station B

$t_1 =$ traveling time from station A to station B

$t_2 =$ traveling time from station D to station B

**Time table:**

$x_3 \geq d_3$

**Continuity:**

$x_3 \geq x_1 + t_1 + w_B$

**Connection:**

$x_3 \geq x_2 + t_2 + c_B$

Departure time of train 1 from station B:

$$x_3 = \max(x_1 + t_1 + w_B, x_2 + t_2 + c_B, d_3)$$
Departure time of train 1 from station B:

\[ x_3 = \max(x_1 + 25, x_2 + 33, 56) \]

\[ = (x_1 \otimes 25) \oplus (x_2 \otimes 33) \oplus 56 \]

\[ = \begin{bmatrix} 25 & 33 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus 56 \]

For a network with departure vector \( x \) we obtain:

\[ x = A \otimes x \oplus d \]

For a cyclic timetable this will result in:

\[ x(k) = A \otimes x(k - 1) \oplus d(k) \]
Legged robots

1 2 3 4 5 6

1 2 3 4 5 6

Aerial phase

Ground phase

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Legged robots

Ground phase

Aerial phase

Ground phase

$t$
Legged robots

Ground phase
Aerial phase

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Legged robots

Ground phase

Aerial phase

Ground phase

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Legged robots

Ground phase

Aerial phase

Ground phase

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Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

- INPUT
- PRINTING
- OUTPUT
- INVERTER
Large-scale Printers
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT

PRINTING

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INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers
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INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

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INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

Diagram showing the process flow of an input, printing, and output system with an inverter.
Large-scale Printers

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22 STEPS: 3 SHEETS

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers
Large-scale Printers

INPUT

PRINTING

OUTPUT

INVERTER
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT

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Large-scale Printers
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

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Large-scale Printers

Diagram showing the process from input to output with an inverter in the middle.
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers

INPUT → PRINTING → INVERTER → OUTPUT
Large-scale Printers

INPUT

PRINTING

INVERTER

OUTPUT
Large-scale Printers
Large-scale Printers

20 STEPS: 6 SHEETS

INPUT

PRINTING

INVERTER

OUTPUT
Container terminal

QUAY CRANES  AGV’S  YARD CRANES

Synchronization crane and vehicle

Synchronization vehicle and stack
quay crane  AGV  yard crane
Synchronization 1
Synchronization 2
Synchronization 3
Max-Plus Linear systems
Max-plus-linear zero-element and unit-element

Max-plus zero: \(-\infty\)

\[ x \oplus (-\infty) = \max(x, -\infty) = x \quad x \otimes (-\infty) = x + (-\infty) = -\infty \]

Max-plus unit: 0

\[ x \otimes 0 = x + 0 = x \]
Max-plus-linear eigenvalues and eigenvectors

For a cyclic timetable the system description is given by

\[ x(k) = A \otimes x(k - 1) \oplus d(k) \]

Max-plus-linear eigenvalue \( \lambda \) and eigenvector \( v \):

\[ A \otimes v = \lambda \otimes v \]

Properties:

\( \lambda \) = natural cycle time (For Dutch railway network: \( \pm 57 \) minutes).

\( v \) = natural timetable (For well-defined network: \( v \approx d \)).
Switching max-plus linear system

- Railway systems: Change order of trains.
- Legged robot: Change gait of robot.
- Paper flow in printers: Change paper size/thickness.
- Container terminal: Change route of container.
- Production system: Choose machine for processing.

The system can operate in a different modes

\[ x(k) = A^{(\ell)}(k) \otimes x(k - 1) \oplus B^{(\ell)}(k) \otimes u(k) \]

in which \( A^{(\ell)} \) and \( B^{(\ell)} \) are system matrices for \( \ell \)-th mode.
Max-min-plus-scaling systems

Piecewise affine systems

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Max-min-plus-scaling (MMPS) systems

System description

\[ x(k+1) = f_x(x(k), u(k)) \]
\[ y(k) = f_y(x(k), u(k)), \]

where entries of \( f_x \) and \( f_y \) are MMPS expressions in \( x(k), u(k) \).

Example MMPS system:

\[ x(k+1) = \max(-2x+5, 3) + \min(x-3, \max(-x+3, 2x-7)) \]
Max-min-plus-scaling system:

\[ x(k + 1) = \max(-2x + 5, 3) + \min(x - 3, \max(-x + 3, x - 7)) \]
Min-max canonical form:

\[ x(k + 1) = \min\left( \max(-x + 2, x), \max(-x + 6, x - 4) \right) \]
Max-min canonical form:

\[ x(k + 1) = \max\left( -x + 2, \min(x, -x + 6), x - 4 \right) \]
Difference canonical form:

\[ x(k + 1) = \max(-2x + 3, 1, 2x - 9) - \max(x - 5, -x + 1) \]
MSc thesis subjects

Max-plus linear systems
- Modeling & control of MPL systems (theory / application)
- Stochastic MPL systems.
- Application using MPL models.

Max-min-plus-scaling linear systems
- Modeling & control of MMPS systems (theory / application)
- Stochastic MMPS systems.
Important courses

- SC42055 - Optimization in Systems & Control
- SC42125 - Model Predictive Control
- SC42075 - Modeling & Control of Hybrid Systems
- WI4062TU - Transport, Routing and Scheduling
- CIE5803-09 - Railway traffic management
Questions ??